EUCLID

BOOKS I-III

HALL & STEVENS

&Mc.

A. W. Lacy

A TEXT-BOOK OF EUCLID'S ELEMENTS

BOOKS I.–III.

A TEXT-BOOK

OF

EUCLID'S ELEMENTS

FOR THE USE OF SCHOOLS

BOOKS I.–III.

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EXTRACT FROM THE PREFACE TO THE FIRST EDITION.

THIS volume contains the first Six Books and part of the Eleventh Book of Euclid's Elements, together with additional Theorems and Examples, giving the most important elementary developments of Euclidean Geometry.

The text has been carefully revised, and special attention given to those points which experience has shewn to present difficulties to beginners.

In the course of this revision the enunciations have been altered as little as possible: and very few departures have been made from Euclid's proofs; in each case changes have been adopted only where the old text has been generally found a cause of difficulty; and such changes are for the most part in favour of well-recognised alternatives.

In Book I., for example, the ambiguity has been removed from the Enunciations of Propositions 18 and 19, and the fact that Propositions 8 and 26 establish the complete equality of the two triangles considered has been strongly urged: thus the redundant step has been removed from Proposition 34.

In Book II. Simson's arrangement of Proposition 13 has been abandoned for a well-known equivalent.

In Book III. Propositions 35 and 36 have been treated generally, and it has not been thought necessary to do more than call attention in a note to the special cases.

These are the chief deviations from the ordinary text as regards method and arrangement of proof; they are points familiar as difficulties to most teachers, and to name them indicates sufficiently, without further enumeration, the general principles which have guided our revision.

A few alternative proofs of difficult propositions are given for the convenience of those teachers who care to use them.

One purpose of the book is *gradually* to familiarise the student with the use of legitimate symbols and abbreviations; for a geometrical argument may thus be thrown into a form which is not only more readily seized by an advanced reader, but is useful as a guide to the way in which Euclid's propositions may be handled in written work. On the other hand, we think it very desirable to defer the introduction of symbols until the beginner has learnt that they can only be properly used in Pure Geometry as abbreviations for verbal argument: and we hope thus to prevent the slovenly and inaccurate habits which are very apt to arise from their employment before this principle is fully recognised.

Accordingly in Book I. we have used no contractions or symbols of any kind, though we have introduced verbal alterations into the text wherever it appeared that conciseness or clearness would be gained.

In Book II. abbreviated forms of constantly recurring words are used, and the phrases *therefore* and *is equal to* are replaced by the usual symbols.

In the Third and following Books, and in additional matter throughout the whole, we have employed all such signs and abbreviations as we believe to add to the clearness of the reasoning, care being taken that the symbols chosen are compatible with a rigorous geometrical method, and are recognised by the majority of teachers.

If this arrangement should be thought fanciful or wanting in uniformity, we may plead that it is the outcome of long experience in the use of various text-books. For some years, for example, we were accustomed to teach from a symbolical text, but in consequence of the frequent misconceptions and inaccuracies which too great brevity was found to generate among beginners, we were compelled to return to one of the older and unabbreviated editions. The gain to our younger boys was immediate and unmistakeable; but the change has not

been unattended with disadvantage to more advanced students, who on reaching the Third or Fourth Book may not only be safely trusted with a carefully chosen system of abbreviations, but are certainly retarded by the monotonous and lengthy formalities of the old text.

It must be understood that our use of symbols, and the removal of unnecessary verbiage and repetition, by no means implies a desire to secure brevity at all hazards. On the contrary, nothing appears to us more mischievous than an abridgement which is attained by omitting steps, or condensing two or more steps into one. Such uses spring from the pressure of examinations; but an examination is not, or ought not to be, a mere race; and while we wish to indicate generally in the later books how a geometrical argument may be abbreviated for the purposes of written work, we have not attempted to reduce the propositions to the barest skeleton which a lenient Examiner may be supposed to accept. Indeed it does not follow that the form most suitable for the page of a text-book is also best adapted to examination purposes; for the object to be attained in each case is entirely different. The text-book should present the argument in the clearest possible manner to the mind of a reader to whom it is new: the written proposition need only convey to the Examiner the assurance that the proposition has been thoroughly grasped and remembered by the pupil.

From first to last we have kept in mind the undoubted fact that a very small proportion of those who study Elementary Geometry, and study it with profit, are destined to become mathematicians in any special sense; and that, to a large majority of students, Euclid is intended to serve not so much as a first lesson in *mathematical* reasoning, as the first, and sometimes the only, model of formal argument presented in an elementary education.

This consideration has determined not only the full treatment of the earlier Books, but the retention of the formal, if somewhat cumbrous, methods of Euclid in many places where proofs of greater brevity and mathematical elegance are available.

We hope that the additional matter introduced into the book will provide sufficient exercise for pupils whose study of Euclid is preliminary to a mathematical education.

The questions distributed through the text follow very easily from the propositions to which they are attached, and we think that teachers are likely to find in them all that is needed for an average pupil reading the subject for the first time.

The Theorems and Examples at the end of each Book contain questions of a slightly more difficult type : they have been very carefully classified and arranged, and brought into close connection with typical examples worked out either partially or in full ; and it is hoped that this section of the book, on which much thought has been expended, will do something towards removing that extreme want of freedom in solving deductions that is so commonly found even among students who have a good knowledge of the text of Euclid.

To Volumes containing Books I-III., and Books I.-IV. an Appendix is added, giving an elementary account of the properties of Pole and Polar, and Radical Axis.

. . . . . .

Throughout the book we have italicised those deductions on which we desired to lay special stress as being in themselves important geometrical results : this arrangement we think will be useful to teachers who have little time to devote to riders, or who wish to sketch out a suitable course for revision.

. . . . . .

H. S. HALL.
F. H. STEVENS.

CLIFTON, *December*, 1886.

PREFATORY NOTE TO THE NEW EDITION.

In the present edition the text has received further revision, and the notes have been for the most part re-written, with a view to greater clearness and simplicity.

References to the *Definitions* being frequent in the text of Book I., the convenience of a *standard order* has been pointed out to us by many elementary teachers. We have therefore thought it advisable to re-number the Definitions in accordance with Simson's edition. This has involved the insertion of certain definitions hitherto omitted as of slight importance: such insertions have now been printed in subordinate type.

A few typographical improvements have been introduced: notably the italicising of *Particular Enunciations*. Some changes in pagination have also been effected for the purpose of presenting the whole of a proposition at one view, or of bringing notes and exercises into closer connection with the text to which they refer. Further, the symbols " $\therefore$ " for *therefore*, and " $=$ " for *is equal to* are now introduced from the 35th Proposition of Book I.

Groups of Test Questions for Revision have been inserted at various stages. These may be useful to beginners, and suggestive to teachers in framing examination papers, which so often consist of mere monotonous lists of propositions and examples.

One important change has been made. The algebraical treatment of the subject-matter of Book V. has been entirely separated from the stricter general treatment, so as to present in the simplest form such Definitions and Theorems of Proportion as are necessary before entering upon Book VI. This Introduction will be found immediately preceding Book VI. in a chapter called *The Elementary Principles of Proportion.*

H. S. H.
F. H. S.

February, 1900.

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EUCLID'S ELEMENTS.

BOOK I.

Definitions.

1. A **point** is that which has position, but no magnitude.

2. A **line** is that which has length without breadth.

3. The extremities of a line are points, and the intersection of two lines is a point.

4. A **straight line** is that which lies evenly between its extreme points.

Any portion cut off from a straight line is called a **segment** of it.

5. A **surface** (or superficies) is that which has length and breadth, but no thickness.

6. The boundaries of a surface are lines.

7. A **plane surface** is one in which any two points being taken, the straight line between them lies wholly in that surface.

A plane surface is frequently referred to simply as a *plane*.

Note. Euclid regards a point merely as a *mark of position*, and he therefore attaches to it no idea of size and shape.

Similarly he considers that the properties of a line arise only from its *length* and *position*, without reference to that minute breadth which every line must really have *if actually drawn*, even though the most perfect instruments are used.

The definition of a surface is to be understood in a similar way.

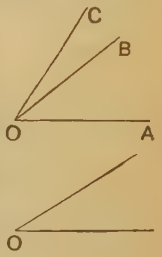
8. A **plane angle** is the inclination of two lines to one another, which meet together, but are not in the same direction.

[Definition 8 is not required in Euclid's Geometry, the only angles employed by him being those formed by *straight* lines. See Def. 9.]

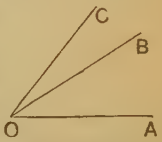
9. A **plane rectilineal angle** is the inclination of two *straight* lines to one another, which meet together, but are not in the same straight line.

The point at which the straight lines meet is called the **vertex** of the angle, and the straight lines themselves the **arms** of the angle.

NOTE. When there are several angles at one point, each is expressed by three letters, of which the letter that refers to the vertex is put between the other two. Thus the angle contained by the straight lines OA, OB is named the angle AOB or BOA; and the angle contained by OA, OC is named the angle AOC or COA. But if there is only one angle at a point, it may be expressed by a single letter, as *the angle at* O.



Of the two straight lines OB, OC shewn in the adjoining diagram, we recognize that OC is *more inclined* than OB to the straight line OA: this we express by saying that the angle AOC is greater than the angle AOB. Thus an angle must be regarded as having *magnitude*.

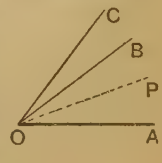


It must be carefully observed that the size of an angle in no way depends on the length of its arms, but only on their *inclination* to one another.

The angle AOC is the *sum* of the angles AOB and BOC; and AOB is the *difference* of the angles AOC and BOC.

[Another view of an angle is recognized in many branches of mathematics; and though not employed by Euclid, it is here given because it furnishes more clearly than any other a conception of what is meant by the *magnitude* of an angle.

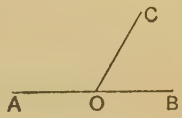
Suppose that the straight line OP in the diagram is capable of revolution about the point O, like the hand of a watch, but in the opposite direction; and suppose that in this way it has passed successively from the position OA to the positions occupied by OB and OC. Such a line must have undergone *more turning* in passing from OA to



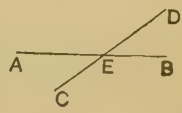
OC, than in passing from OA to OB ; and consequently the angle AOC is said to be greater than the angle AOB.]

Angles which lie on either side of a common arm are called **adjacent** angles.

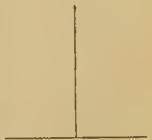
For example, when one straight line OC is drawn from a point in another straight line AB, the angles COA, COB are *adjacent*.



When two straight lines, such as AB, CD, cross one another at E, the two angles CEA, BED are said to be **vertically opposite**. The two angles CEB, AED are also vertically opposite to one another.



10. When a straight line standing on another straight line makes the adjacent angles equal to one another, each of the angles is called a **right angle** ; and the straight line which stands on the other is called a **perpendicular** to it.

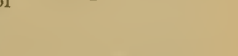


11. An **obtuse angle** is an angle which is greater than a right angle.

12. An **acute angle** is an angle which is less than a right angle.

[In the adjoining figure the straight line OB may be supposed to have arrived at its present position, from the position occupied by OA, by revolution about the point O in *either* of the two directions indicated by the arrows : thus two straight lines drawn from a point may be considered as forming *two* angles (marked (i) and (ii) in the figure), of which the greater (ii) is said to be **reflex**.

If the arms OA, OB are in the same straight line, the angle formed by them on either side is called a **straight angle**.]



13. A term or boundary is the extremity of anything.

14. Any portion of a plane surface bounded by one or more lines is called a **plane figure.**

The sum of the bounding lines is called the **perimeter** of the figure.

Two figures are said to be equal in **area** when they enclose equal portions of a plane surface.

15. A **circle** is a plane figure contained by one line, which is called the **circumference**, and is such that all straight lines drawn from a certain point within the figure to the circumference are equal to one another; this point is called the **centre** of the circle.

16. A **radius** of a circle is a straight line drawn from the centre to the circumference.

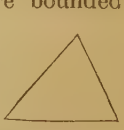
17. A **diameter** of a circle is a straight line drawn through the centre, and terminated both ways by the circumference.

18. A **semicircle** is the figure bounded by a diameter of a circle and the part of the circumference cut off by the diameter.

19. A **segment of a circle** is the figure bounded by a straight line and the part of the circumference which it cuts off.

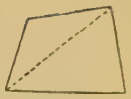
20. **Rectilineal figures** are those which are bounded by straight lines.

21. A **triangle** is a plane figure bounded by *three* straight lines.



Any one of the angular points of a triangle may be regarded as its **vertex**; and the opposite side is then called the **base.**

22. A **quadrilateral** is a plane figure bounded by *four* straight lines.



The straight line which joins opposite angular points in a quadrilateral is called a **diagonal**.

23. A **polygon** is a plane figure bounded by more than four straight lines.

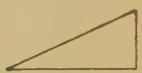
TRIANGLES.

24. An **equilateral triangle** is a triangle whose three sides are equal.

25. An **isosceles triangle** is a triangle two of whose sides are equal.

26. A **scalene triangle** is a triangle which has three unequal sides.

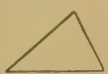
27. A **right-angled triangle** is a triangle which has a right angle.



The side opposite to the right angle in a right-angled triangle is called the **hypotenuse**.

28. An **obtuse-angled triangle** is a triangle which has an obtuse angle.

29. An **acute-angled triangle** is a triangle which has *three* acute angles.



[It will be seen hereafter (Book I. Proposition 17) *that every triangle must have at least two acute angles.*]

QUADRILATERALS.

30. A **square** is a four-sided figure which has all its sides equal and all its angles right angles.

[It may be shewn that if a quadrilateral has all its sides equal and *one* angle a right angle, then *all* its angles will be right angles.]

31. An **oblong** is a four-sided figure which has all its angles right angles, but not all its sides equal.

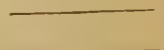
32. A **rhombus** is a four-sided figure which has all its sides equal, but its angles are not right angles.

33. A **rhomboid** is a four-sided figure which has its opposite sides equal to one another, but all its sides are not equal nor its angles right angles.

34. All other four-sided figures are called **trapeziums.**

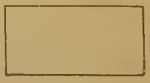
It is usual now to restrict the term *trapezium* to a quadrilateral which has two of its sides *parallel.* [See Def. 35.]

35. **Parallel straight lines** are such as, being in the same plane, do not meet, however far they are produced in either direction.



36. A **Parallelogram** is a four-sided figure which has its opposite sides parallel.

37. A **rectangle** is a parallelogram which has one of its angles a right angle.



The Postulates.

Let it be granted,

1. *That a straight line may be drawn from any one point to any other point.*

2. *That a* finite, *that is to say a terminated, straight line may be* produced *to any length in that straight line.*

3. *That a circle may be described from any centre, at any distance from that centre, that is, with a radius equal to any finite straight line drawn from the centre.*

Notes on the Postulates.

1. In order to draw the diagrams required in Euclid's Geometry certain instruments are necessary. These are

(i) A *ruler* with which to draw straight lines.

(ii) A *pair of compasses* with which to draw circles.

In the *Postulates*, or requests, Euclid claims the use of these instruments, and assumes that they suffice for the purposes mentioned above.

2. It is important to notice that the Postulates include no means of *direct measurement*: hence the straight ruler is not supposed to be *graduated*; and the compasses are not to be employed for *transferring distances* from one part of a diagram to another.

3. When we draw a straight line from the point A to the point B, we are said to *join* AB.

To *produce* a straight line means to *prolong* or *lengthen* it.

The expression *to describe* is used in Geometry in the sense of *to draw*.

On the Axioms.

The science of Geometry is based upon certain simple statements, the truth of which is so evident that they are accepted without proof.

These self-evident truths, called by Euclid *Common Notions*, are known as the **Axioms.**

General Axioms.

1. *Things which are equal to the same thing are equal to one another.*

2. *If equals be added to equals, the wholes are equal.*

3. *If equals be taken from equals, the remainders are equal.*

4. *If equals be added to unequals, the wholes are unequal, the greater sum being that which includes the greater of the unequals.*

5. *If equals be taken from unequals, the remainders are unequal, the greater remainder being that which is left from the greater of the unequals.*

6. *Things which are double of the same thing, or of equal things, are equal to one another.*

7. *Things which are halves of the same thing, or of equal things, are equal to one another.*

9.* *The whole is greater than its part.*

*To preserve the classification of general and geometrical axioms, we have placed Euclid's *ninth* axiom before the *eighth*.

Geometrical Axioms.

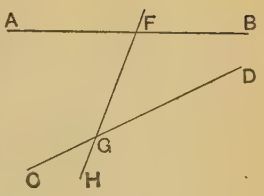
8. *Magnitudes which can be made to coincide with one another, are equal.*

10. *Two straight lines cannot enclose a space.*

11. *All right angles are equal.*

12. *If a straight line meet two straight lines so as to make the interior angles on one side of it together less than two right angles, these straight lines will meet if continually produced on the side on which are the angles which are together less than two right angles.*

That is to say, if the two straight lines AB and CD are met by the straight line EH at F and G, in such a way that the angles BFG, DGF are together less than two right angles, it is asserted that AB and CD will meet if continually produced in the direction of B and D.



NOTES ON THE AXIOMS.

1. The necessary characteristics of an Axiom are

(i) That it should be *self-evident*; that is, that its truth should be immediately accepted without proof.

(ii) That it should be *fundamental*; that is, that its truth should not be derivable from any other truth more simple than itself.

(iii) That it should supply a basis for the establishment of further truths.

These characteristics may be summed up in the following definition.

DEFINITION. An **Axiom** is a self-evident truth, which neither requires nor is capable of proof, but which serves as a foundation for future reasoning.

2. Euclid's Axioms may be classified as *general* and *geometrical*.

General Axioms apply to *magnitudes of all kinds*. Geometrical Axioms refer specially to *geometrical magnitudes*, as lines, angles, and figures.

3. Axiom 8 is Euclid's test of the equality of two geometrical magnitudes. It implies that any line, angle, or figure, may be taken up from its position, and without change in size or form, laid down upon a second line, angle, or figure, for the purpose of comparison, and it states that two such magnitudes are equal when one can be exactly placed over the other without overlapping.

This process is called **superposition**, and the first magnitude is said to be **applied** to the other.

4. Axiom 12 has been objected to on the double ground that it cannot be considered self-evident, and that its truth may be deduced from simpler principles. It is employed for the first time in the 29th Proposition of Book I., where a short discussion of the difficulty will be found.

INTRODUCTORY.

1. Little is known of Euclid beyond the fact that he lived about three centuries before Christ (325-285) at Alexandria, where he became famous as a writer and teacher of Mathematics.

Among the works ascribed to him, the best known and most important is *The Elements*, written in Greek, and consisting of Thirteen Books. Of these it is now usual to read Books I.-IV. and VI. (which deal with Plane Geometry), together with parts of Books XI. and XII. (on the Geometry of Solids). The remaining Books deal with subjects which belong to the theory of Arithmetic.

2. Plane Geometry deals with the properties of all lines and figures that may be drawn upon a plane surface.

Euclid in his first Six Books confines himself to the properties of straight lines, rectilineal figures, and circles.

3. The subject is divided into a number of separate discussions, called **propositions.**

Propositions are of two kinds, **Problems** and **Theorems.**

A **Problem** proposes to perform some geometrical construction, such as to draw some particular line, or to construct some required figure.

A **Theorem** proposes to prove the truth of some geometrical statement.

4. A Proposition consists of the following parts :

The *General Enunciation*, the *Particular Enunciation*, the *Construction*, and the *Proof.*

(i) The **General Enunciation** is a preliminary statement, describing in general terms the purpose of the proposition.

(ii) The **Particular Enunciation** repeats in special terms the statement already made, and refers it to a diagram, which enables the reader to follow the reasoning more easily.

(iii) The **Construction** then directs the drawing of such straight lines and circles as may be required to effect the purpose of a problem, or to prove the truth of a theorem.

(iv) The **Proof** shews that the object proposed in a problem has been accomplished, or that the property stated in a theorem is true.

5. Euclid's reasoning is said to be **Deductive**, because by a connected chain of argument it *deduces* new truths from truths already proved or admitted. Thus each proposition, though in one sense complete in itself, is derived from the Postulates, Axioms, or former propositions, and itself leads up to subsequent propositions.

6. The initial letters Q.E.F., placed at the end of a problem, stand for **Quod erat Faciendum**, *which was to be done.*

The letters Q.E.D. are appended to a theorem, and stand for **Quod erat Demonstrandum**, *which was to be proved.*

7. A **Corollary** is a statement the truth of which follows readily from an established proposition; it is therefore appended to the proposition as an inference or deduction, which usually requires no further proof.

8. The attention of the beginner is drawn to the special use of the *future tense* in the Particular Enunciations of Euclid's propositions.

The future is only used in a statement of which the truth is *about to be proved.* Thus: "*The triangle* ABC SHALL BE *equilateral*" means that the triangle *has yet to be proved* equilateral. While, "*The triangle* ABC IS *equilateral*" means that the triangle *has already been proved* (or given) equilateral.

9. The following symbols and abbreviations may be employed in writing out the propositions of Book I., though their use is not recommended to beginners.

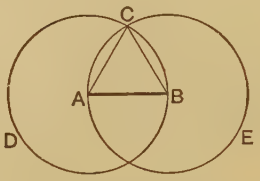
$\therefore$	*for* therefore,	par^{l} (or $\parallel$)	*for* parallel,
$=$	,, is, or are, equal to,	par^{m}	,, parallelogram,
$\angle$	,, angle,	sq.	,, square,
rt. $\angle$	,, right angle,	rectil.	,, rectilineal,
$\triangle$	,, triangle,	st. line	,, straight line,
perp.	,, perpendicular,	pt.	,, point;

and all obvious contractions of words, such as opp., adj., diag., etc., for opposite, adjacent, diagonal, etc.

SECTION I.

PROPOSITION 1. PROBLEM.

To describe an equilateral triangle on a given finite straight line.



Let AB be the given straight line.

It is required to describe an equilateral triangle on AB.

Construction. With centre A, and radius AB, describe the circle BCD. *Post.* 3.

With centre B, and radius BA, describe the circle ACE. *Post.* 3.

From the point C at which the circles cut one another, draw the straight lines CA and CB to the points A and B. *Post.* 1.

Then shall the triangle ABC *be equilateral.*

Proof. Because A is the centre of the circle BCD,
therefore AC is equal to AB. *Def.* 15.

And because B is the centre of the circle ACE,
therefore BC is equal to AB. *Def.* 15.

Therefore AC and BC are each equal to AB.

But things which are equal to the same thing are equal to one another. *Ax.* 1.

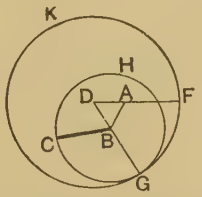
Therefore AC is equal to BC.

Therefore AC, AB, BC are equal to one another.

Therefore the triangle ABC is equilateral;
and it is described on the given straight line AB. Q.E.F.

Proposition 2. Problem.

From a given point to draw a straight line equal to a given straight line.



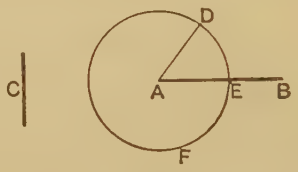
Let A be the given point, and BC the given straight line.
It is required to draw from A *a straight line equal to* BC.

Construction. Join AB; *Post.* 1.
and on AB describe an equilateral triangle DAB. I. 1.
With centre B, and radius BC, describe the circle CGH. *Post.* 3.
Produce DB to meet the circle CGH at G. *Post.* 2.
With centre D, and radius DG, describe the circle GKF.
Produce DA to meet the circle GKF at F. *Post.* 2.
Then AF *shall be equal to* BC.

Proof. Because B is the centre of the circle CGH,
therefore BC is equal to BG. *Def.* 15.
And because D is the centre of the circle GKF,
therefore DF is equal to DG. *Def.* 15.
And DA, a part of DF, is equal to DB, a part of DG; *Def.* 24.
therefore the remainder AF is equal to the remainder BG. *Ax.* 3.
But BC has been proved equal to BG;
therefore AF and BC are each equal to BG.
And things which are equal to the same thing are equal to one another. *Ax.* 1.
Therefore AF is equal to BC;
and it has been drawn from the given point A. Q.E.F.

Proposition 3. Problem.

From the greater of two given straight lines to cut off a part equal to the less.



Let AB and C be the two given straight lines, of which AB is the greater.

It is required to cut off from AB *a part equal to* C.

Construction. From the point A draw the straight line AD equal to C; I. 2.
and with centre A and radius AD, describe the circle DEF, cutting AB at E. *Post* 3.

Then AE *shall be equal to* C.

Proof. Because A is the centre of the circle DEF,
therefore AE is equal to AD. *Def.* 15.
But C is equal to AD. *Constr.*
Therefore AE and C are each equal to AD.
Therefore AE is equal to C; *Ax.* 1.
and it has been cut off from the given straight line AB.

Q.E.F.

EXERCISES ON PROPOSITIONS 1 TO 3.

1. If the two circles in Proposition 1 cut one another again at F, prove that AFB is an equilateral triangle.

2. If the two circles in Proposition 1 cut one another at C and F, prove that the figure ACBF is a rhombus.

3. AB is a straight line of given length: shew how to draw from A a line double the length of AB.

4. Two circles are drawn with the same centre O, and two radii OA, OB are drawn in the smaller circle. If OA, OB are produced to cut the outer circle at D and E, prove that AD = BE.

5. AB is a straight line, and P, Q are two points, one on each side of AB. Shew how to find points in AB, whose distance from P is equal to PQ. How many such points will there be?

6. In the figure of Proposition 2, if AB is equal to BC, shew that D, the vertex of the equilateral triangle, will fall on the circumference of the circle CGH.

7. In Proposition 2 the point A may be joined to either extremity of BC. Draw the figure, and prove the proposition in the case when A is joined to C.

8. On a given straight line AB describe an isosceles triangle having each of its equal sides equal to a given straight line PQ.

9. On a given base describe an isosceles triangle having each of its equal sides double of the base.

10. In a given straight line the points A, M, N, B are taken in order. On AB describe a triangle ABC, such that the side AC may be equal to AN, and the side BC to BM.

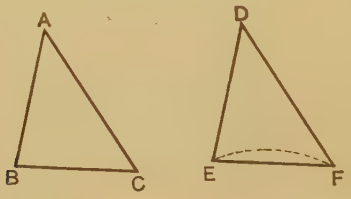
NOTE ON PROPOSITIONS 2 AND 3.

Propositions 2 and 3 are rendered necessary by the restriction tacitly imposed by Euclid, that compasses shall not be used to *transfer distances*. [See Notes on the Postulates.]

In carrying out the construction of Prop. 2 the point A may be joined to *either* extremity of the line BC; the equilateral triangle may be described on *either* side of the line so drawn; and the sides of the equilateral triangle may be produced in *either* direction. Thus there are in general $2 \times 2 \times 2$, or *eight*, possible constructions. The student should exercise himself in drawing the various figures that may arise.

Proposition 4. Theorem.

If two triangles have two sides of the one equal to two sides of the other, each to each, and have also the angles contained by those sides equal, then the triangles shall be equal in all respects; that is to say, their bases or third sides shall be equal, and their remaining angles shall be equal, each to each, namely those to which the equal sides are opposite; and the triangles shall be equal in area.



Let ABC, DEF be two triangles, in which
the side AB is equal to the side DE,
the side AC is equal to the side DF, and
the contained angle BAC is equal to the contained angle EDF.

Then (i) *the base* BC *shall be equal to the base* EF;
(ii) *the angle* ABC *shall be equal to the angle* DEF;
(iii) *the angle* ACB *shall be equal to the angle* DFE;
(iv) *the triangle* ABC *shall be equal to the triangle* DEF *in area.*

Proof. If the triangle ABC be applied to the triangle DEF,
so that the point A may lie on the point D,
and the straight line AB along the straight line DE;
then because AB is equal to DE. *Hyp.*
therefore the point B must coincide with the point E.
And because AB falls along DE,
and the angle BAC is equal to the angle EDF, *Hyp.*
therefore AC must fall along DF.
And because AC is equal to DF, *Hyp.*
therefore the point C must coincide with the point F.
Then since B coincides with E, and C with F,
therefore the base BC must coincide with the base EF;

for if not, two straight lines would enclose a space; which is impossible. *Ax.* 10.

Thus the base BC coincides with the base EF, and is therefore equal to it. *Ax.* 8.

And the remaining angles of the triangle ABC coincide with the remaining angles of the triangle DEF, and are therefore equal to them;

namely, the angle ABC is equal to the angle DEF,

and the angle ACB is equal to the angle DFE.

And the triangle ABC coincides with the triangle DEF, and is therefore equal to it in area. *Ax.* 8.

That is, the triangles are equal in all respects. Q.E.D.

NOTE. The sides and angles of a triangle are known as its *six parts*. A triangle may also be considered in regard to its *area*.

Two triangles are said to be **equal in all respects**, or **identically equal**, when the sides and angles of one are respectively equal to the sides and angles of the other. We have seen that such triangles may be made to *coincide* with one another by *superposition*, so that they are also equal in *area*. [See Note on Axiom 8.]

[It will be shewn later that triangles can be equal in *area* without being equal in their several parts; that is to say, triangles can have the same *area* without having the same *shape*.]

EXERCISES ON PROPOSITION 4.

1. ABCD is a square: prove that the diagonals AC, BD are equal to one another.

2. ABCD is a square, and L, M, and N are the middle points of AB, BC, and CD: prove that

(i) LM = MN. (ii) AM = DM.

(iii) AN = AM. (iv) BN = DM.

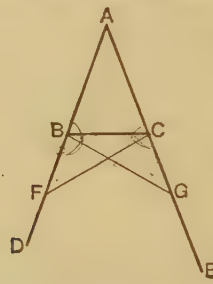
[Draw a separate figure in each case.]

3. ABC is an isosceles triangle: from the equal sides AB, AC two equal parts AX, AY are cut off, and BY and CX are joined. Prove that BY = CX.

4. ABCD is a quadrilateral having the opposite sides BC, AD equal, and also the angle BCD equal to the angle ADC: prove that BD is equal to AC.

Proposition 5. Theorem.

The angles at the base of an isosceles triangle are equal to one another; and if the equal sides be produced, the angles on the other side of the base shall also be equal to one another.



Let ABC be an isosceles triangle, in which
the side AB is equal to the side AC,
and let the straight lines AB, AC be produced to D and E.

Then (i) *the angle* ABC *shall be equal to the angle* ACB;
(ii) *the angle* CBD *shall be equal to the angle* BCE.

Construction. In BD take any point F;
and from AE cut off a part AG equal to AF. I. 3.
Join FC, GB.

Proof. Then in the triangles FAC, GAB,

Because { FA is equal to GA, *Constr.*
and AC is equal to AB, *Hyp.*
also the contained angle at A is common to the two triangles:

therefore the triangle FAC is equal to the triangle GAB in all respects; I. 4.
that is, the base FC is equal to the base GB,
and the angle ACF is equal to the angle ABG,
also the angle AFC is equal to the angle AGB.

Again, because AF is equal to AG,
and AB, a part of AF, is equal to AC, a part of AG; *Hyp.*
therefore the remainder BF is equal to the remainder CG.

Then in the two triangles BFC, CGB,

Because { BF is equal to CG, *Proved.*
and FC is equal to GB, *Proved.*
also the contained angle BFC is equal to the contained angle CGB, *Proved.*

therefore the triangle BFC is equal to the triangle CGB in all respects; I. 4.

so that the angle FBC is equal to the angle GCB, and the angle BCF to the angle CBG.

Now it has been shewn that the angle ABG is equal to the angle ACF,

and that the angle CBG, a part of ABG, is equal to the angle BCF, a part of ACF;

therefore the remaining angle ABC is equal to the remaining angle ACB; *Ax.* 3.

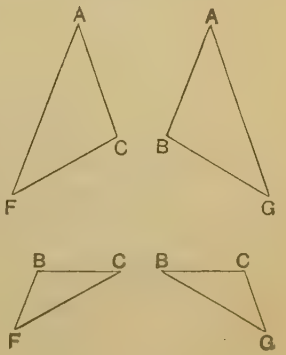
and these are the angles at the base of the triangle ABC.

Also it has been shewn that the angle FBC is equal to the angle GCB;

and these are the angles on the other side of the base. Q.E.D.

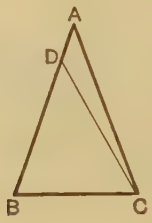
COROLLARY. *Hence if a triangle is equilateral it is also equiangular.*

NOTE. The difficulty which beginners find with this proposition arises from the fact that the triangles to be compared overlap one another in the diagram. This difficulty may be diminished by detaching each pair of triangles from the rest of the figure, as shewn in the margin.



Proposition 6. Theorem.

If two angles of a triangle be equal to one another, then the sides also which subtend, or are opposite to, the equal angles, shall be equal to one another.



Let ABC be a triangle, in which
the angle ABC is equal to the angle ACB.
Then shall the side AC *be equal to the side* AB.

Construction. For if AC be not equal to AB,
one of them must be greater than the other.
If possible, let AB be the greater ;
and from it cut off BD equal to AC. I. 3.
Join DC.

Proof. Then in the triangles DBC, ACB,

Because { DB is equal to AC, *Constr.*
and BC is common to both,
also the contained angle DBC is equal to the contained angle ACB ; *Hyp.*

therefore the triangle DBC is equal to the triangle ACB in area, I. 4.
the part equal to the whole ; which is absurd. *Ax.* 9.

Therefore AB is not unequal to AC ;
that is, AB is equal to AC. Q.E.D.

Corollary. *Hence if a triangle is equiangular it is also equilateral.*

NOTE ON PROPOSITIONS 5 AND 6.

The enunciation of a theorem consists of two clauses. The first clause tells us what we are to *assume*, and is called the **hypothesis**; the second tells us what *it is required to prove*, and is called the **conclusion**.

For example, the enunciation of Proposition 5 assumes that in a certain triangle ABC *the side* AB = *the side* AC: this is the *hypothesis*. From this it is required to prove that *the angle* ABC = *the angle* ACB: this is the *conclusion*.

If we interchange the hypothesis and conclusion of a theorem, we enunciate a new theorem which is called the **converse** of the first.

For example, in Prop. 5

it is *assumed* that AB = AC;
it is *required to prove* that the angle ABC = the angle ACB.

Now in Prop. 6

it is *assumed* that the angle ABC = the angle ACB;
it is *required to prove* that AB = AC.

Thus we see that Prop. 6 is the converse of Prop. 5; for *the hypothesis of each is the conclusion of the other*.

In Proposition 6 Euclid employs for the first time an *indirect method of proof* frequently used in geometry. It consists in shewing that the theorem *cannot be untrue*; since, if it were, we should be led to some *impossible conclusion*. This form of proof is known as **Reductio ad Absurdum**, and is most commonly used in demonstrating the converse of some foregoing theorem.

The converse of *all* true theorems are not themselves necessarily true. [See Note on Prop 8.]

EXERCISES ON PROPOSITION 5.

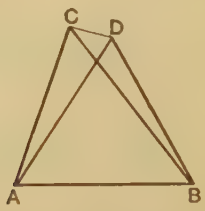
1. ABCD is a rhombus, in which the diagonal BD is drawn: shew that
   (i) the angle ABD = the angle ADB;
   (ii) the angle CBD = the angle CDB;
   (iii) the angle ABC = the angle ADC.

2. ABC, DBC are two isosceles triangles drawn on the same base BC, but on *opposite* sides of it: prove (by means of I. 5) that the angle ABD = the angle ACD.

3. ABC, DBC are two isosceles triangles drawn on the same base BC and on *the same* side of it: employ I. 5 to prove that the angle ABD = the angle ACD.

Proposition 7. Theorem.

On the same base, and on the same side of it, there cannot be two triangles having their sides which are terminated at one extremity of the base equal to one another, and likewise those which are terminated at the other extremity equal to one another.



If it be possible, on the same base AB, and on the same side of it, let there be two triangles ACB, ADB in which
the side AC is equal to the side AD,
and also the side BC is equal to the side BD.

Case I. When the vertex of each triangle is without the other triangle.

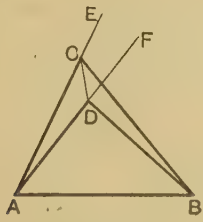
Construction. Join CD.

Proof. Then in the triangle ACD,
because AC is equal to AD, *Hyp.*
therefore the angle ACD is equal to the angle ADC. I. 5.

But the whole angle ACD is greater than its part, the angle BCD;
therefore also the angle ADC is greater than the angle BCD;
still more then is the angle BDC greater than the angle BCD.

Again, in the triangle BCD,
because BC is equal to BD, *Hyp.*
therefore the angle BDC is equal to the angle BCD: I. 5.
but it was shewn to be greater; which is impossible.

CASE II. When one of the vertices, as D, is within the other triangle ACB.



Construction. As before, join CD;
and produce AC, AD to E and F.

Proof. Then in the triangle ACD,
because AC is equal to AD, *Hyp.*
therefore the angle ECD is equal to the angle FDC,
these being the angles on the other side of the base. I. 5.
But the angle ECD is greater than its part, the angle BCD;
therefore the angle FDC is also greater than the angle BCD:
still more then is the angle BDC greater than the angle BCD.

Again, in the triangle BCD,
because BC is equal to BD, *Hyp.*
therefore the angle BDC is equal to the angle BCD: I. 5.
but it has been shewn to be greater; which is impossible.

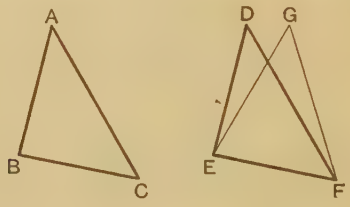
The case in which the vertex of one triangle is on a side of the other needs no demonstration.

Therefore AC cannot be equal to AD, and *at the same time*, BC equal to BD. Q.E.D.

NOTE. The sides AC, AD are called **conterminous** sides; similarly the sides BC, BD are conterminous.

Proposition 8. Theorem.

If two triangles have two sides of the one equal to two sides of the other, each to each, and have likewise their bases equal, then the angle which is contained by the two sides of the one shall be equal to the angle which is contained by the two sides of the other



Let ABC, DEF be two triangles, in which
the side AB is equal to the side DE,
the side AC is equal to the side DF,
and the base BC is equal to the base EF.

Then shall the angle BAC *be equal to the angle* EDF.

Proof. If the triangle ABC be applied to the triangle DEF,
so that the point B falls on the point E,
and the base BC along the base EF;
then because BC is equal to EF, *Hyp.*
therefore the point C must coincide with the point F.

Then since BC coincides with EF,
it follows that BA and AC must coincide with ED and DF:
for if they did not, but took some other position, as EG, GF,
then on the same base EF, and on the same side of it, there would be two triangles EDF, EGF, having their *conterminous* sides equal: namely ED equal to EG, and FD equal to FG
But this is impossible. I. 7.

Therefore the sides BA, AC coincide with the sides ED, DF.
That is, the angle BAC coincides with the angle EDF, and is
therefore equal to it. *Ax.* 8.

Q.E.D.

NOTE 1. In this Proposition the three sides of one triangle are given equal respectively to the three sides of the other ; and from this it is shewn that the two triangles may be made *to coincide with one another.*

Hence we are led to the following important Corollary.

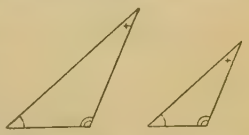
COROLLARY. *If in two triangles the three sides of the one are equal to the three sides of the other, each to each, then the triangles are equal in all respects.*

[An alternative proof, which is independent of Prop. 7, will be found on page 26.]

NOTE 2. Proposition 8 furnishes an instance of a true theorem of which the *converse* is not necessarily true.

It is proved above that *if the sides of one triangle are severally equal to the sides of another, then the angles of the first triangle are severally equal to the angles of the second.*

The *converse* of this enunciation would be as follows: *If the angles of one triangle are severally equal to the angles of another, then the sides of the first triangle are equal to the sides of the second.*



But this, as the diagram in the margin shews, is by no means necessarily true.

EXERCISES ON PROPOSITION 8.

1. Shew (by drawing a diagonal) that the opposite angles of a rhombus are equal.

2. If ABCD is a quadrilateral, in which AB = CD and AD = CB, prove that the angle ADC = the angle ABC.

3. If ABC and DBC are two isosceles triangles drawn on the same base BC, prove (by means of I. 8) that the angle ABD = the angle ACD, taking (i) the case where the triangles are on the *same* side of BC, (ii) the case where they are on *opposite* sides of BC.

4. If ABC, DBC are two isosceles triangles drawn on opposite sides of the same base BC, and if AD be joined, prove that each of the angles BAC, BDC will be divided into two equal parts.

5. If in the figure of Ex. 4 the line AD meets BC in E, prove that BE = EC.

Proposition 8. Alternative Proof.

Let ABC and DEF be two triangles, which have the sides BA, AC equal respectively to the sides ED, DF, and the base BC equal to the base EF.

Then shall the angle BAC *be equal to the angle* EDF.

For apply the triangle ABC to the triangle DEF, so that B may fall on E, and BC along EF, and so that the point A may be on the side of EF remote from D;

then C must fall on F, since BC is equal to EF.

Let GEF be the new position of the triangle ABC.

Join DG.

Case I. When DG intersects EF.

Then because ED = EG,

∴ the angle EDG = the angle EGD. I. 5.

Again because FD = FG,

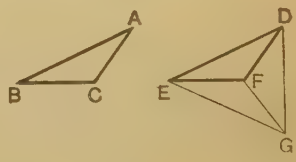
∴ the angle FDG = the angle FGD. I. 5.

Hence the whole angle EDF = the whole angle EGF; *Ax.* 2.

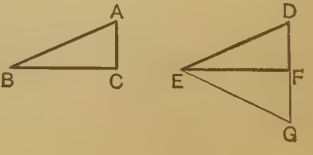
that is, the angle EDF = the angle BAC.

Two cases remain which may be dealt with in a similar manner: namely,

Case II. When DG meets EF produced.



Case III. When one pair of sides, as DF, FG are in one straight line.



QUESTIONS AND EXERCISES FOR REVISION.

1. Define *adjacent angles, a right angle, vertically opposite angles.*

2. Explain the words *enunciation, hypothesis, conclusion.*

3. Distinguish between the meanings of the following statements:

(i) then AB *is* equal to PQ;

(ii) then AB *shall be* equal to PQ.

4. When are two theorems said to be *converse* to one another. Give an example.

5. Shew by an example that the converse of a true theorem is not itself necessarily true.

6. What is a *corollary*? Quote the corollary to Proposition 5; and shew how its truth follows from that proposition.

7. Name the six *parts* of a triangle. When are triangles said to be *equal in all respects*?

8. What do you understand by the expression *geometrical magnitudes*? Give examples?

9. What is meant by *superposition*? Explain the test by which Euclid determines if two geometrical magnitudes are equal to one another. Illustrate by an example.

10. Quote and explain the *third postulate.* What restrictions does Euclid impose on the use of compasses, and what problems are thereby made necessary?

11. Define an *axiom.* Quote the axioms referred to (i) in Proposition 2; (ii) in Proposition 7.

12. Prove by the method of *superposition* that two squares are equal in area, if a side of one is equal to a side of the other.

13. Two quadrilaterals ABCD, EFGH have the sides AB, BC, CD, DA equal respectively to the sides EF, FG, GH, HE, and have also the angle BAD equal to the angle FEH. Shew that the figures may be made to coincide with one another.

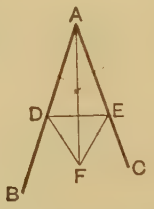
14. AB, AC are the equal sides of an isosceles triangle ABC; and L, M, N are the middle points of AB, BC, and CA respectively: prove that

(i) LM = MN. (ii) BN = CL.

(iii) the angle ALM = the angle ANM.

Proposition 9. Problem.

To bisect a given rectilineal angle, that is, to divide it into two equal parts.



Let BAC be the given angle.
It is required to bisect the angle BAC.

Construction. In AB take any point D;
and from AC cut off AE equal to AD. I. 3.
Join DE;
and on DE, on the side remote from A, describe an equilateral triangle DEF. I. 1.
Join AF.
Then shall the straight line AF *bisect the angle* BAC.

Proof. For in the two triangles DAF, EAF,

Because { DA is equal to EA, *Constr.*
and AF is common to both;
and the third side DF is equal to the third side EF; *Def.* 24.

therefore the angle DAF is equal to the angle EAF. I. 8.
Therefore the given angle BAC is bisected by the straight line AF. Q.E.F.

EXERCISES.

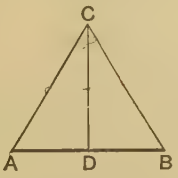
1. If in the above figure the equilateral triangle DFE were described on the same side of DE as A, what different cases would arise? And under what circumstances would the construction fail?

2. In the same figure, shew that AF also bisects the angle DFE.

3. **Divide an angle into four equal parts.**

PROPOSITION 10. PROBLEM.

To bisect a given finite straight line, that is, to divide it into two equal parts.



Let AB be the given straight line.
It is required to divide AB *into two equal parts.*

Constr. On AB describe an equilateral triangle ABC; I. 1.
and bisect the angle ACB by the straight line CD, meeting AB at D. I. 9.

Then shall AB *be bisected at the point* D.

Proof. For in the triangles ACD, BCD,

Because { AC is equal to BC, *Def.* 24.
and CD is common to both;
also the contained angle ACD is equal to the contained angle BCD; *Constr.*

therefore the triangle ACD is equal to the triangle BCD in all respects: I. 4.

so that the base AD is equal to the base BD.

Therefore the straight line AB is bisected at the point D.

Q.E.F.

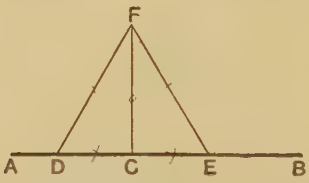
EXERCISES.

1. Shew that the straight line which bisects the vertical angle of an isosceles triangle, also bisects the base.

2. On a given base describe an isosceles triangle such that the sum of its equal sides may be equal to a given straight line.

Proposition 11. Problem.

To draw a straight line at right angles to a given straight line, from a given point in the same.



Let AB be the given straight line, and C the given point in it.

It is required to draw from C *a straight line at right angles to* AB.

Construction. In AC take any point D,
and from CB cut off CE equal to CD. I. 3.
On DE describe the equilateral triangle DFE. I. 1.
Join CF.
Then shall CF *be at right angles to* AB.

Proof. For in the triangles DCF, ECF,
Because { DC is equal to EC, *Constr.*
and CF is common to both;
and the third side DF is equal to the third side EF: *Def.* 24.
therefore the angle DCF is equal to the angle ECF: I. 8.
and these are adjacent angles.

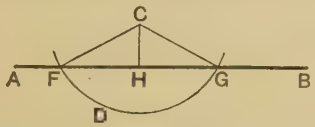
But when one straight line, standing on another, makes the adjacent angles equal, each of these angles is called a right angle; *Def.* 10.
therefore each of the angles DCF, ECF is a right angle.
Therefore CF is at right angles to AB,
and has been drawn from a point C in it. Q.E.F.

EXERCISE.

In the figure of the above proposition, shew that *any* point in FC, or FC produced, is equidistant from D and E.

PROPOSITION 12. PROBLEM.

To draw a straight line perpendicular to a given straight line of unlimited length, from a given point without it.



Let AB be the given straight line of unlimited length, and let C be the given point without it.

It is required to draw from C *a straight line perpendicular to* AB.

Construction. On the side of AB remote from C take any point D;

and with centre C, and radius CD, describe the circle FDG, cutting AB at F and G.

Bisect FG at H; 1. 10.

and join CH.

Then shall CH *be perpendicular to* AB.

Join CF and CG.

Proof. Then in the triangles FHC, GHC,

Because { FH is equal to GH, *Constr.*
and HC is common to both;
and the third side CF is equal to the third side CG, being radii of the circle FDG; *Def.* 15.

therefore the angle CHF is equal to the angle CHG; I. 8.

and these are adjacent angles.

But when one straight line, standing on another, makes the adjacent angles equal, each of these angles is called a right angle, and the straight line which stands on the other is called a perpendicular to it. *Def.* 10.

Therefore CH is perpendicular to AB,

and has been drawn from the point C without it. Q.E.F.

NOTE. The line AB must be of unlimited length, that is, capable of production to an indefinite length in either direction, to ensure its being intersected in two points by the circle FDG.

QUESTIONS AND EXERCISES FOR REVISION.

1. Distinguish between a *problem* and a *theorem*.

2. When are two figures said to be *identically equal*? Under what conditions has it so far been proved that two *triangles* are identically equal?

3. Explain the method of proof known as *Reductio ad Absurdum*. Quote the enunciations of the propositions in which this method has so far been used.

4. Quote the corollaries of Propositions 5 and 6, and shew that each is the converse of the other.

5. What is meant by saying that Euclid's reasoning is *deductive*? Shew, for instance, that the proof of Proposition 5 is a deductive argument.

6. Two forts defend the mouth of a river, one on each side; the forts are 4000 yards apart, and their guns have a range of 3000 yards. Taking *one inch* to represent a length of 1000 yards, draw a diagram shewing what part of the river is exposed to the fire of both forts.

7. Define *the perimeter* of a rectilineal figure. A square and an equilateral triangle each have a perimeter of 3 feet: compare the lengths of their sides.

8. Shew how to draw a rhombus each of whose sides is equal to a given straight line PQ, which is also to be one diagonal of the figure.

9. A and B are two given points. Shew how to draw a rhombus having A and B as opposite vertices, and having each side equal to a given line PQ. Is this always possible?

10. Two circles are described with the same centre O; and two radii OA, OB are drawn to the inner circle, and produced to cut the outer circle at D and E: prove that

(i) DB = EA;

(ii) the angle BAD = the angle ABE;

(iii) the angle ODB = the angle OEA.

EXERCISES ON PROPOSITIONS 1 TO 12.

1. Shew that the straight line which joins the vertex of an isosceles triangle to the middle point of the base is perpendicular to the base.

2. Shew that the straight lines which join the extremities of the base of an isosceles triangle to the middle points of the opposite sides, are equal to one another.

3. Two given points in the base of an isosceles triangle are equidistant from the extremities of the base : shew that they are also equidistant from the vertex.

4. If the opposite sides of a quadrilateral are equal, shew that the opposite angles are also equal.

5. Any two isosceles triangles XAB, YAB stand on the same base AB : shew that the angle XAY is equal to the angle XBY ; and if XY be joined, that the angle AXY is equal to the angle BXY.

6. Shew that the opposite angles of a rhombus are bisected by the diagonal which joins them.

7. Shew that the straight lines which bisect the base angles of an isosceles triangle form with the base a triangle which is also isosceles.

8. ABC is an isosceles triangle having AB equal to AC ; and the angles at B and C are bisected by straight lines which meet at O : shew that OA bisects the angle BAC.

9. Shew that the triangle formed by joining the middle points of the sides of an equilateral triangle is also equilateral.

10. The equal sides BA, CA of an isosceles triangle BAC are produced beyond the vertex A to the points E and F, so that AE is equal to AF ; and FB, EC are joined : shew that FB is equal to EC.

11. Shew that the diagonals of a rhombus bisect one another at right angles.

12. In the equal sides AB, AC of an isosceles triangle ABC two points X and Y are taken, so that AX is equal to AY ; and CX and BY are drawn intersecting in O : shew that

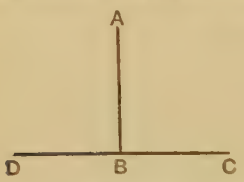
(i) the triangle BOC is isosceles ;
(ii) AO bisects the vertical angle BAC ;
(iii) AO, if produced, bisects BC at right angles.

13. Describe an isosceles triangle, having given the base and the length of the perpendicular drawn from the vertex to the base.

14. In a given straight line find a point that is equidistant from two given points. In what case is this impossible ?

Proposition 13. Theorem.

The adjacent angles which one straight line makes with another straight line, on one side of it, are either two right angles or are together equal to two right angles.



Let the straight line AB meet the straight line DC.

Then the adjacent angles DBA, ABC *shall be either two right angles, or together equal to two right angles.*

Case I. For if the angle DBA is equal to the angle ABC, each of them is a right angle. *Def.* 10.

Case II. But if the angle DBA is not equal to the angle ABC,

from B draw BE at right angles to CD. I. 11.

Proof. Now the angle DBA is made up of the two angles DBE, EBA;

to each of these equals add the angle ABC:

then the two angles DBA, ABC are together equal to the three angles DBE, EBA, ABC. *Ax.* 2.

Again, the angle EBC is made up of the two angles EBA, ABC;

to each of these equals add the angle DBE:

then the two angles DBE, EBC are together equal to the three angles DBE, EBA, ABC. *Ax.* 2.

But the two angles DBA, ABC have been shewn to be equal to the same three angles;

therefore the angles DBA, ABC are together equal to the angles DBE, EBC. *Ax.* 1.

But the angles DBE, EBC are two right angles: *Constr.*

therefore the angles DBA, ABC are together equal to two right angles. Q.E.D.

DEFINITIONS.

(i) The **complement** of an acute angle is its *defect from* a right angle, that is, the angle by which it falls short of a right angle.

Thus two angles are **complementary,** when their sum is a right angle.

(ii) The **supplement** of an angle is its *defect from* two right angles, that is, the angle by which it falls short of two right angles.

Thus two angles are **supplementary,** when their sum is two right angles.

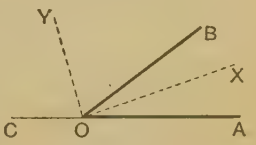
COROLLARY. *Angles which are complementary or supplementary to the same angle are equal to one another.*

EXERCISES.

1. If the two exterior angles formed by producing a side of a triangle both ways are equal, shew that the triangle is isosceles.

2. *The bisectors of the adjacent angles which one straight line makes with another contain a right angle.*

NOTE In the adjoining diagram AOB is a given angle; and one of its arms AO is produced to C: the adjacent angles AOB, BOC are bisected by OX, OY.



Then OX and OY are called respectively the **internal** and **external bisectors** of the angle AOB.

Hence Exercise 2 may be thus enunciated:

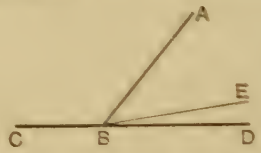
The internal and external bisectors of an angle are at right angles to one another.

3. Shew that the angles AOX and COY are complementary.

4. Shew that the angles BOX and COX are supplementary; and also that the angles AOY and BOY are supplementary.

Proposition 14. Theorem.

If, at a point in a straight line, two other straight lines, on opposite sides of it, make the adjacent angles together equal to two right angles, then these two straight lines shall be in one and the same straight line.



At the point B in the straight line AB, let the two straight lines BC, BD, on the opposite sides of AB, make the adjacent angles ABC, ABD together equal to two right angles.

Then BD *shall be in the same straight line with* BC.

Proof. For if BD be not in the same straight line with BC, if possible, let BE be in the same straight line with BC.

Then because AB meets the straight line CBE,
therefore the adjacent angles CBA, ABE are together equal to two right angles. I. 13.
But the angles CBA, ABD are also together equal to two right angles. *Hyp.*
Therefore the angles CBA, ABE are together equal to the angles CBA, ABD. *Ax.* 11.
From each of these equals take the common angle CBA:
then the remaining angle ABE is equal to the remaining angle ABD: the part equal to the whole: which is impossible.
Therefore BE is not in the same straight line with BC.
And in the same way it may be shewn that no other line but BD can be in the same straight line with BC.
Therefore BD is in the same straight line with BC. Q.E.D.

EXERCISE.

ABCD is a rhombus: and the diagonal AC is bisected at O. If O is joined to the angular points B and D; shew that OB and OD are in one straight line.

PROPOSITION 15. THEOREM.

If two straight lines intersect one another, then the vertically opposite angles shall be equal.

Let the two straight lines AB, CD cut one another at the point E.

Then (i) *the angle* AEC *shall be equal to the angle* DEB;
(ii) *the angle* CEB *shall be equal to the angle* AED.

Proof. Because AE meets the straight line CD,
therefore the adjacent angles CEA, AED are together equal to two right angles. I. 13.

Again, because DE meets the straight line AB,
therefore the adjacent angles AED, DEB are together equal to two right angles. I. 13.

Therefore the angles CEA, AED are together equal to the angles AED, DEB.

From each of these equals take the common angle AED;
then the remaining angle CEA is equal to the remaining angle DEB. *Ax.* 3.

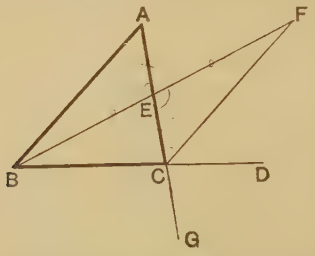
In the same way it may be proved that the angle CEB is equal to the angle AED. Q.E.D.

COROLLARY 1. *From this it follows that, if two straight lines cut one another, the four angles so formed are together equal to four right angles.*

COROLLARY 2. *Consequently, when any number of straight lines meet at a point, the sum of the angles made by consecutive lines is equal to four right angles.*

Proposition 16. Theorem.

If one side of a triangle be produced, then the exterior angle shall be greater than either of the interior opposite angles.



Let ABC be a triangle, and let BC be produced to D.

Then shall the exterior angle ACD *be greater than either of the interior opposite angles* ABC, BAC.

Construction. Bisect AC at E; I. 10.
Join BE; and produce it to F, making EF equal to BE. I. 3.
Join FC.

Proof. Then in the triangles AEB, CEF,

Because { AE is equal to CE, *Constr.*
and EB is equal to EF; *Constr.*
also the angle AEB is equal to the vertically opposite angle CEF; I. 15.

therefore the triangle AEB is equal to the triangle CEF in all respects: I. 4.
so that the angle BAE is equal to the angle ECF.

But the angle ECD is greater than its part, the angle ECF;
therefore the angle ECD is greater than the angle BAE;
that is, the angle ACD is greater than the angle BAC.

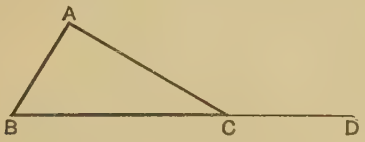
In the same way, if BC be bisected, and the side AC produced to G, it may be proved that the angle BCG is greater than the angle ABC.

But the angle BCG is equal to the angle ACD: I. 15
therefore also the angle ACD is greater than the angle ABC.

Q.E.D.

PROPOSITION 17. THEOREM.

Any two angles of a triangle are together less than two right angles.



Let ABC be a triangle.

Then shall any two of the angles of the triangle ABC *be together less than two right angles.*

Construction. Produce the side BC to D.

Proof. Then because BC, a side of the triangle ABC, is produced to D;

therefore the exterior angle ACD is greater than the interior opposite angle ABC. I. 16.

To each of these add the angle ACB:

then the angles ACD, ACB are together greater than the angles ABC, ACB. *Ax.* 4.

But the adjacent angles ACD, ACB are together equal to two right angles. I. 13.

Therefore the angles ABC, ACB are together less than two right angles.

Similarly it may be shewn that the angles BAC, ACB, as also the angles CAB, ABC, are together less than two right angles. Q.E.D.

NOTE. It follows from this Proposition that *every triangle must have at least two acute angles*: for if one angle is obtuse, or a right angle, each of the other angles must be less than a right angle.

EXERCISES.

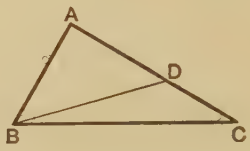
1. Enunciate this Proposition so as to shew that it is the converse of Axiom 12.

2. If any side of a triangle is produced both ways, the exterior angles so formed are together greater than two right angles.

3. Shew how a proof of Proposition 17 may be obtained by joining each vertex in turn to any point in the opposite side.

Proposition 18. Theorem.

If one side of a triangle be greater than another, then the angle opposite to the greater side shall be greater than the angle opposite to the less.



Let ABC be a triangle, in which the side AC is greater than the side AB.

Then shall the angle ABC *be greater than the angle* ACB.

Construction. From AC cut off a part AD equal to AB. I. 3.
Join BD.

Proof. Then in the triangle ABD,
because AB is equal to AD,
therefore the angle ABD is equal to the angle ADB. I. 5.

But the exterior angle ADB of the triangle DCB is greater than the interior opposite angle DCB, that is, greater than the angle ACB. I. 16.

Therefore also the angle ABD is greater than the angle ACB; still more then is the angle ABC greater than the angle ACB. Q.E.D.

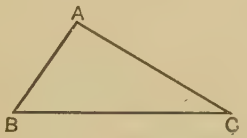
Euclid enunciated Proposition 18 as follows:

The greater side of every triangle has the greater angle opposite to it.

[This form of enunciation is found to be a common source of difficulty with beginners, who fail to distinguish what is *assumed* in it and what is *to be proved*. If Euclid's enunciations of Props. 18 and 19 are adopted, it is important to remember that in each case the part of the triangle *first named* points out the hypothesis.]

PROPOSITION 19. THEOREM.

If one angle of a triangle be greater than another, then the side opposite to the greater angle shall be greater than the side opposite to the less.



Let ABC be a triangle in which the angle ABC is greater than the angle ACB.

Then shall the side AC *be greater than the side* AB.

Proof. For if AC be not greater than AB,
it must be either equal to, or less than AB.

But AC is not equal to AB,
for then the angle ABC would be equal to the angle ACB; I. 5.
but it is not. *Hyp.*

Neither is AC less than AB;
for then the angle ABC would be less than the angle ACB; I. 18.
but it is not. *Hyp.*

That is, AC is neither equal to, nor less than AB.
Therefore AC is greater than AB. Q.E.D.

NOTE. The mode of demonstration used in this Proposition is known as the **Proof by Exhaustion.** It is applicable to cases in which *one* of certain suppositions must necessarily be true: and it consists in shewing that each of these suppositions is false *with one exception*: hence the truth of the remaining supposition is inferred.

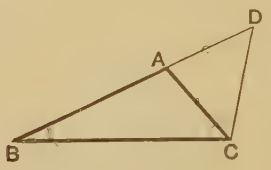
Euclid enunciated Proposition 19 as follows:

The greater angle of every triangle is subtended by the greater side, or, has the greater side opposite to it.

[For Exercises on Props. 18 and 19 see page 44.]

Proposition 20. Theorem.

Any two sides of a triangle are together greater than the third side.



Let ABC be a triangle.

Then shall any two of its sides be together greater than the third side:

namely, BA, AC, *shall be greater than* CB;
AC, CB *shall be greater than* BA;
and CB, BA *shall be greater than* AC.

Construction. Produce BA to D, making AD equal to AC. I. 3.
Join DC.

Proof. Then in the triangle ADC,
because AD is equal to AC, *Constr.*
therefore the angle ACD is equal to the angle ADC. I. 5.
But the angle BCD is greater than its part the angle ACD;
therefore also the angle BCD is greater than the angle ADC,
that is, than the angle BDC.

And in the triangle BCD,
because the angle BCD is greater than the angle BDC,
therefore the side BD is greater than the side CB. I. 19.

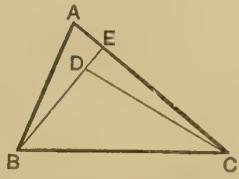
But BA and AC are together equal to BD:
therefore BA and AC are together greater than CB.

Similarly it may be shewn
that AC, CB are together greater than BA;
and CB, BA are together greater than AC. Q.E.D.

[For Exercises see page 44.]

PROPOSITION 21. THEOREM.

If from the ends of a side of a triangle, there be drawn two straight lines to a point within the triangle, then these straight lines shall be less than the other two sides of the triangle, but shall contain a greater angle.



Let ABC be a triangle, and from B, C, the ends of the side BC, let the straight lines BD, CD be drawn to a point D within the triangle

Then (i) BD *and* DC *shall be together less than* BA *and* AC ;

(ii) *the angle* BDC *shall be greater than the angle* BAC.

Construction. Produce BD to meet AC in E.

Proof. (i) In the triangle BAE, the two sides BA, AE are together greater than the third side BE ; I. 20.

to each of these add EC ;

then BA, AC are together greater than BE, EC. *Ax.* 4.

Again, in the triangle DEC, the two sides DE, EC are together greater than DC ; I. 20.

to each of these add BD ;

then BE, EC are together greater than BD, DC.

But it has been shewn that BA, AC are together greater than BE, EC :

still more then are BA, AC greater than BD, DC.

(ii) Again, the exterior angle BDC of the triangle DEC is greater than the interior opposite angle DEC ; I. 16.

and the exterior angle DEC of the triangle BAE is greater than the interior opposite angle BAE, that is, than the angle BAC ; I. 16.

still more then is the angle BDC greater than the angle BAC.

Q.E.D.

EXERCISES.

ON PROPOSITIONS 18 AND 19.

1. The hypotenuse is the greatest side of a right-angled triangle.

2. If two angles of a triangle are equal to one another, the sides also, which subtend the equal angles, are equal to one another. Prove this [*i.e.* Prop. 6] indirectly by using the result of Prop. 18.

3. BC, the base of an isosceles triangle ABC, is produced to any point D; shew that AD is greater than either of the equal sides.

4. If in a quadrilateral the greatest and least sides are opposite to one another, then each of the angles adjacent to the least side is greater than its opposite angle.

5. In a triangle ABC, if AC is not greater than AB, shew that any straight line drawn through the vertex A and terminated by the base BC, is less than AB.

6. ABC is a triangle, in which OB, OC bisect the angles ABC, ACB respectively: shew that, if AB is greater than AC, then OB is greater than OC.

ON PROPOSITION 20.

7. The difference of any two sides of a triangle is less than the third side.

8. In a quadrilateral, if two opposite sides which are not parallel are produced to meet one another; shew that the perimeter of the greater of the two triangles so formed is greater than the perimeter of the quadrilateral.

9. The sum of the distances of any point from the three angular points of a triangle is greater than half its perimeter.

10. The perimeter of a quadrilateral is greater than the sum of its diagonals.

11. Obtain a proof of Proposition 20 by bisecting an angle by a straight line which meets the opposite side.

ON PROPOSITION 21.

12. In Proposition 21 shew that the angle BDC is greater than the angle BAC by joining AD, and producing it towards the base.

13. The sum of the distances of any point within a triangle from its angular points is less than the perimeter of the triangle.

QUESTIONS FOR REVISION.

1. Define the *complement* of an angle. When are two angles said to be *supplementary*? Shew that two angles which are supplementary to the same angle are equal to one another.

2. What is meant by an angle being *bisected internally and externally*?

Prove that the internal and external bisectors of an angle are at right angles to one another.

3. Prove that the sum of the angles formed by any number of straight lines drawn from a point is equal to four right angles.

4. Why must every triangle have *at least two acute angles*? Quote the enunciation of the proposition from which this inference is drawn.

5. In the enunciation *The greater side of a triangle has the greater angle opposite to it*, point out what is assumed and what is to be proved.

6. What is meant by the *Proof by Exhaustion*? Illustrate the use of this method by naming the steps in the proof of Proposition 19.

7. What inference may be drawn respecting the triangles whose sides measure

(i) 4 inches, 5 inches, 4 inches;
(ii) 8 inches, 9 inches, 10 inches;
(iii) 6 inches, 10 inches, 4 inches?

8. Quote the enunciations of propositions which, from a hypothesis relating to the *sides* of triangle, establish a conclusion relating to the *angles*.

9. Quote the enunciations of propositions which, from a hypothesis relating to the *angles* of a triangle, establish a conclusion relating to the sides.

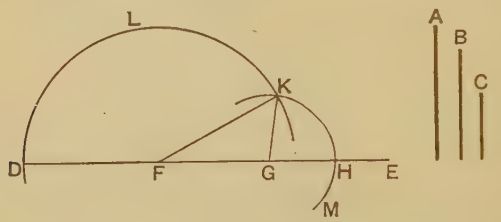
10. Explain why parallel straight lines must be *in the same plane*.

11. Prove by means of Prop. 7 that on a given base and on the same side of it only one equilateral triangle can be drawn.

12. In an isosceles triangle, if the equal sides are produced, shew that the angles on the other side of the base must be obtuse.

Proposition 22. Problem.

To describe a triangle having its sides equal to three given straight lines, any two of which are together greater than the third.



Let A, B, C be the three given straight lines, of which any two are together greater than the third.

It is required to describe a triangle of which the sides shall be equal to A, B, C.

Construction. Take a straight line DE terminated at the point D, but unlimited towards E.

Make DF equal to A, FG equal to B, and GH equal to C. I. 3.

With centre F and radius FD, describe the circle DLK.

With centre G and radius GH, describe the circle MHK cutting the former circle at K.

Join FK, GK.

Then shall the triangle KFG *have its sides equal to the three straight lines* A, B, C.

Proof. Because F is the centre of the circle DLK,

therefore FK is equal to FD : *Def.* 15.

but FD is equal to A ; *Constr.*

therefore also FK is equal to A. *Ax.* 1.

Again, because G is the centre of the circle MHK,

therefore GK is equal to GH : *Def.* 15.

but GH is equal to C ; *Constr.*

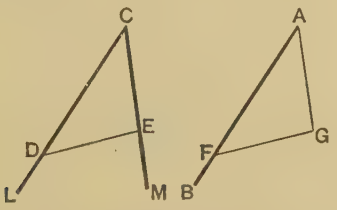
therefore also GK is equal to C. *Ax.* 1.

And FG is equal to B. *Constr.*

Therefore the triangle KFG has its sides KF, FG, GK equal respectively to the three given lines A, B, C. Q.E.F.

PROPOSITION 23. PROBLEM.

At a given point in a given straight line, to make an angle equal to a given rectilineal angle.



Let AB be the given straight line, and A the given point in it, and let LCM be the given angle.

It is required to draw from A *a straight line making with* AB *an angle equal to the given angle* DCE.

Construction. In CL, CM take any points D and E; and join DE.

From AB cut off AF equal to CD. I. 3.

On AF describe the triangle FAG, having the remaining sides AG, GF equal respectively to CE, ED. I. 22.

Then shall the angle FAG *be equal to the angle* DCE.

Proof. For in the triangles FAG, DCE,

Because { FA is equal to DC, *Constr.*
and AG is equal to CE; *Constr.*
and the base FG is equal to the base DE: *Constr.*

therefore the angle FAG is equal to the angle DCE. I. 8.

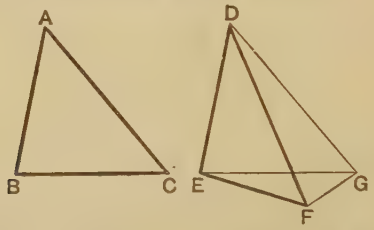
That is, AG makes with AB, at the given point A, an angle equal to the given angle DCE. Q.E.F.

EXERCISE.

On a given base describe a triangle, whose remaining sides shall be equal to two given straight lines. Point out how the construction fails, if any one of the three given lines is greater than the sum of the other two.

Proposition 24. Theorem.

If two triangles have two sides of the one equal to two sides of the other, each to each, but the angle contained by the two sides of one greater than the angle contained by the corresponding sides of the other; then the base of that which has the greater angle shall be greater than the base of the other.



Let ABC, DEF be two triangles, in which
the side BA is equal to the side ED,
and the side AC is equal to the side DF,
but the angle BAC is greater than the angle EDF.

Then shall the base BC *be greater than the base* EF.

Of the two sides DE, DF, let DE be that which is not greater than the other.*

Construction. At D in the straight line ED, and on the same side of it as DF, make the angle EDG equal to the angle BAC. I. 23.

Make DG equal to DF or AC; I. 3.
and join EG, GF.

Proof. Then in the triangles BAC, EDG,

Because { BA is equal to ED, *Hyp.*
and AC is equal to DG, *Constr.*
also the contained angle BAC is equal to the contained angle EDG; *Constr.*

therefore the triangle BAC is equal to the triangle EDG in all respects: I. 4.

so that the base BC is equal to the base EG.

Again, in the triangle FDG,
because DG is equal to DF,
therefore the angle DFG is equal to the angle DGF. I. 5.

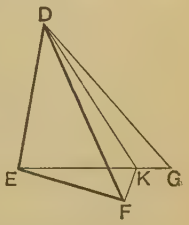
But the angle DGF is greater than its part the angle EGF; therefore also the angle DFG is greater than the angle EGF; still more then is the angle EFG greater than the angle EGF.

And in the triangle EFG,
because the angle EFG is greater than the angle EGF,
therefore the side EG is greater than the side EF; I. 19.
but EG was shewn to be equal to BC;
therefore BC is greater than EF. Q.E.D.

*The object of this step is to make the point F fall *below* EG. Otherwise F might fall *above*, *upon*, or *below* EG; and each case would require separate treatment. But as it is not *proved* that this condition fulfils its object, this demonstration of Prop. 24 must be considered defective.

An alternative construction and proof are given below.

Construction. At D in ED make the angle EDG equal to the angle BAC; and make DG equal to DF. Join EG.



Then, as before, it may be shewn that the triangle EDG = the triangle BAC in all respects.

Now if EG passes through F, then EG is greater than EF; that is, BC is greater than EF.

But if not, bisect the angle FDG by DK, meeting EG at K. Join FK.

Proof. Then in the triangles FDK, GDK,

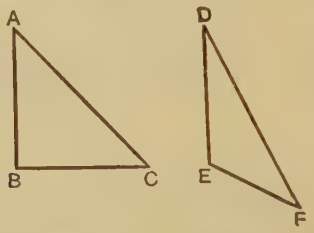
Because { FD = GD,
and DK is common to both,
and the angle FDK = the angle GDK; *Constr.*

∴ FK = GK. I. 4.

But in the triangle EKF, the two sides EK, KF are greater than EF; that is, EK, KG are greater than EF.
Hence EG (or BC) is greater than EF.

PROPOSITION 25. THEOREM.

If two triangles have two sides of the one equal to two sides of the other, each to each, but the base of one greater than the base of the other; then the angle contained by the sides of that which has the greater base, shall be greater than the angle contained by the corresponding sides of the other.



Let ABC, DEF be two triangles in which
the side BA is equal to the side ED,
and the side AC is equal to the side DF,
but the base BC is greater than the base EF.
Then shall the angle BAC *be greater than the angle* EDF.

Proof. For if the angle BAC be not greater than the angle EDF, it must be either equal to, or less than the angle EDF.

But the angle BAC is not equal to the angle EDF,
for then the base BC would be equal to the base EF; I. 4.
but it is not. *Hyp.*

Neither is the angle BAC less than the angle EDF,
for then the base BC would be less than the base EF; I. 24.
but it is not. *Hyp.*

Therefore the angle BAC is neither equal to, nor less than the angle EDF;
that is, the angle BAC is greater than the angle EDF. Q.E.D.

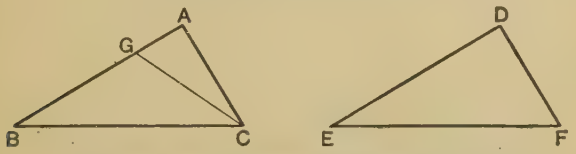
EXERCISE.

In a triangle ABC, the vertex A is joined to X, the middle point of the base BC; shew that the angle AXB is obtuse or acute, according as AB is greater or less than AC.

PROPOSITION 26. THEOREM.

If two triangles have two angles of the one equal to two angles of the other, each to each, and a side of one equal to a side of the other, these sides being either adjacent to the equal angles, or opposite to equal angles in each; then shall the triangles be equal in all respects.

CASE I. When the equal sides are *adjacent* to the equal angles in the two triangles.



Let ABC, DEF be two triangles, in which
the angle ABC is equal to the angle DEF,
and the angle ACB is equal to the angle DFE,
and the side BC is equal to the side EF.

Then shall the triangle ABC *be equal to the triangle* DEF *in all respects; that is,* AB *shall be equal to* DE, *and* AC *to* DF, *and the angle* BAC *shall be equal to the angle* EDF.

For if AB be not equal to DE, one must be greater than the other. If possible, let AB be greater than DE.

Construction. From BA cut off BG equal to ED, and join GC. I. 3.

Proof. Then in the two triangles GBC, DEF,

Because { GB is equal to DE, *Constr.*
and BC is equal to EF, *Hyp.*
also the contained angle GBC is equal to the contained angle DEF; *Hyp.*

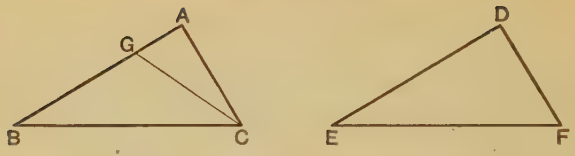
therefore the triangle GBC is equal to the triangle DEF in all respects; I. 4.

so that the angle GCB is equal to the angle DFE.

But the angle ACB is equal to the angle DFE; *Hyp.*

therefore also the angle GCB is equal to the angle ACB; *Ax.* 1.

the part equal to the whole, which is impossible.



Therefore AB is not unequal to DE;
that is, AB is equal to DE.

Hence in the triangles ABC, DEF,

Because { AB is equal to DE, *Proved.*
and BC is equal to EF; *Hyp.*
also the contained angle ABC is equal to the contained angle DEF; *Hyp.*

therefore the triangle ABC is equal to the triangle DEF in all respects: I. 4.

so that the side AC is equal to the side DF;
and the angle BAC is equal to the angle EDF.

Q.E.D.

CASE II. When the equal sides are *opposite* to equal angles in the two triangles.

Let ABC, DEF be two triangles, in which
the angle ABC is equal to the angle DEF,
and the angle ACB is equal to the angle DFE,
and the side AB is equal to the side DE.

Then the triangle ABC *shall be equal to the triangle* DEF *in all respects;*

namely, BC *shall be equal to* EF,
and AC *shall be equal to* DF,
and the angle BAC *shall be equal to the angle* EDF.

For if BC be not equal to EF, one must be greater than the other. If possible, let BC be greater than EF.

Construction. From BC cut off BH equal to EF, I. 3.
and join AH.

Proof. Then in the triangles ABH, DEF,

Because { AB is equal to DE, *Hyp.*
and BH is equal to EF, *Constr.*
also the contained angle ABH is equal to the contained angle DEF; *Hyp.*

therefore the triangle ABH is equal to the triangle DEF in all respects; I. 4.

so that the angle AHB is equal to the angle DFE.

But the angle DFE is equal to the angle ACB; *Hyp.*

therefore the angle AHB is equal to the angle ACB; *Ax.* 1.

that is, an exterior angle of the triangle ACH is equal to an interior opposite angle; which is impossible. I. 16.

Therefore BC is not unequal to EF,
that is, BC is equal to EF.

Hence in the triangles ABC, DEF,

Because { AB is equal to DE, *Hyp.*
and BC is equal to EF; *Proved.*
also the contained angle ABC is equal to the contained angle DEF; *Hyp.*

therefore the triangle ABC is equal to the triangle DEF in all respects; I. 4.

so that the side AC is equal to the side DF,
and the angle BAC is equal to the angle EDF.

Q.E.D.

COROLLARY. *In both cases of this Proposition it is seen that the triangles may be made* to coincide with one another; *and they are therefore equal* in area.

ON THE IDENTICAL EQUALITY OF TRIANGLES.

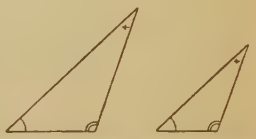
Three cases have been already dealt with in Propositions 4, 8, and 26, the results of which may be summarized as follows :

Two triangles are equal in all respects when the following three parts in each are severally equal :

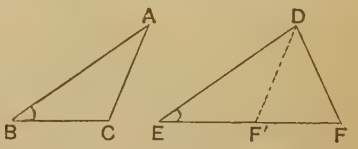
1. Two sides, and the included angle. *Prop.* 4.
2. The three sides. *Prop.* 8, *Cor.*
3. (*a*) Two angles, and the adjacent side ; (*b*) Two angles, and a side opposite one of them. } *Prop.* 26.

Two triangles are not, however, necessarily equal in all respects when *any three parts* of one are equal to the corresponding parts of the other. For example

(i) When the *three angles* of one are equal to the *three angles* of the other, each to each, the adjoining diagram shews that the triangles need not be equal in all respects.



(ii) When *two sides and one angle* in one are equal to *two sides and one angle* in the other, the given angles being *opposite* to equal sides, the diagram shews that the triangles need not be equal in all respects.



For it will be seen that if AB = DE, and AC = DF, and the angle ABC = the angle DEF, then the shorter of the given sides in the triangle DEF may lie in either of the positions DF or DF'.

In cases (i) and (ii) a further condition must be given before we can prove that the two triangles are identically equal.

We observe that in each of the three cases in which two triangles have been proved equal in all respects, namely in Propositions 4, 8, 26, it is shewn that the triangles may be made to *coincide with one another* ; so that they are equal in *area*. Euclid however restricted himself to the use of Prop. 4, when he required to deduce the equality in *area* of two triangles from the equality of certain of their parts. This restriction is now generally abandoned.

EXERCISES ON PROPOSITIONS 12–26.

1. If BX and CY, the bisectors of the angles at the base BC of an isosceles triangle ABC, meet the opposite sides in X and Y, shew that the triangles YBC, XCB are equal in all respects.

2. Shew that the perpendiculars drawn from the extremities of the base of an isosceles triangle to the opposite sides are equal.

3. *Any point on the bisector of an angle is equidistant from the arms of the angle.*

4. Through O, the middle point of a straight line AB, any straight line is drawn, and perpendiculars AX and BY are dropped upon it from A and B : shew that AX is equal to BY.

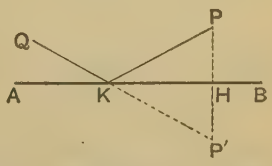
5. If the bisector of the vertical angle of a triangle is at right angles to the base, the triangle is isosceles.

6. *The perpendicular is the shortest straight line that can be drawn from a given point to a given straight line ; and of others, that which is nearer to the perpendicular is less than the more remote ; and two, and only two equal straight lines can be drawn from the given point to the given straight line, one on each side of the perpendicular.*

7. *From two given points on the same side of a given straight line, draw two straight lines, which shall meet in the given straight line, and make equal angles with it.*

Let AB be the given straight line, and P, Q the given points.

It is required to draw from P and Q to a point in AB, two straight lines that shall be equally inclined to AB.



Construction. From P draw PH perpendicular to AB : produce PH to P', making HP' equal to PH. Draw QP', meeting AB in K. Join PK.

Then PK, QK shall be the required lines. [Supply the proof.]

8. In a given straight line find a point which is equidistant from two given intersecting straight lines. In what case is this impossible?

9. Through a given point draw a straight line such that the perpendiculars drawn to it from two given points may be equal.

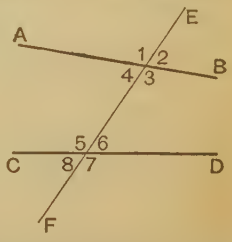
In what case is this impossible?

SECTION II.

PARALLEL STRAIGHT LINES AND PARALLELOGRAMS.

DEFINITION. Parallel straight lines are such as, being in the same plane, do not meet however far they are produced in both directions.

When two straight lines AB, CD are met by a third straight line EF, *eight* angles are formed, to which for the sake of distinction particular names are given.



Thus in the adjoining figure,

1, 2, 7, 8 are called **exterior** angles,

3, 4, 5, 6 are called **interior** angles,

4 and 6 are said to be **alternate** angles;

so also the angles 3 and 5 are alternate to one another.

Of the angles 2 and 6, 2 is referred to as the exterior angle, and 6 as the **interior opposite** angle on the same side of EF.

2 and 6 are sometimes called **corresponding** angles.

So also, 1 and 5, 7 and 3, 8 and 4 are corresponding angles.

Euclid's treatment of parallel straight lines is based upon his twelfth Axiom, which we here repeat.

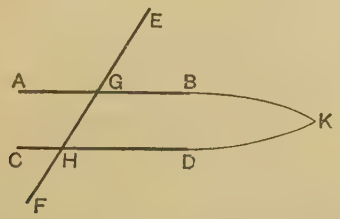
AXIOM 12. If a straight line cut two straight lines so as to make the two interior angles on the same side of it together less than two right angles, these straight lines, being continually produced, will at length meet on that side on which are the angles which are together less than two right angles.

Thus in the figure given above, if the two angles 3 and 6 are together less than two right angles, it is asserted that AB and CD will meet towards B and D.

This Axiom is used to establish I. 29: some remarks upon it will be found in a note on that Proposition.

Proposition 27. Theorem.

If a straight line, falling on two other straight lines, make the alternate angles equal to one another, then these two straight lines shall be parallel.



Let the straight line EF cut the two straight lines AB, CD at G and H, so as to make the alternate angles AGH, GHD equal to one another.

Then shall AB *and* CD *be parallel.*

Proof. For if AB and CD be not parallel,

they will meet, if produced, either towards B and D, or towards A and C.

If possible, let AB and CD, when produced, meet towards B and D, at the point K.

Then KGH is a triangle, of which one side KG is produced to A;

therefore the exterior angle AGH is greater than the interior opposite angle GHK. I. 16.

But the angle AGH was given equal to the angle GHK: *Hyp.*

hence the angles AGH and GHK are both equal and unequal;

which is impossible.

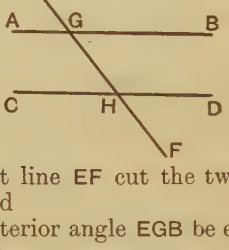
Therefore AB and CD cannot meet when produced towards B and D.

Similarly it may be shewn that they cannot meet towards A and C:

therefore AB and CD are parallel.

Proposition 28. Theorem.

If a straight line, falling on two other straight lines, make an exterior angle equal to the interior opposite angle on the same side of the line; or if it make the interior angles on the same side together equal to two right angles, then the two straight lines shall be parallel.



Let the straight line EF cut the two straight lines AB, CD in G and H: and

First, let the exterior angle EGB be equal to the interior opposite angle GHD.

Then shall AB *and* CD *be parallel.*

Proof. Because the angle EGB is equal to the angle GHD; and because the angle EGB is also equal to the vertically opposite angle AGH; I. 15.

therefore the angle AGH is equal to the angle GHD;

but these are alternate angles;

therefore AB and CD are parallel. I. 27.

Q.E.D.

Secondly, let the two interior angles BGH, GHD be together equal to two right angles.

Then shall AB *and* CD *be parallel.*

Proof. Because the angles BGH, GHD are together equal to two right angles; *Hyp.*

and because the adjacent angles BGH, AGH are also together equal to two right angles; I. 13.

therefore the angles BGH, AGH are together equal to the two angles BGH, GHD.

From these equals take the common angle BGH:

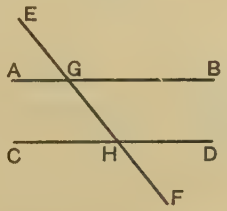
then the remaining angle AGH is equal to the remaining angle GHD: and these are alternate angles;

therefore AB and CD are parallel. I. 27.

Q.E.D.

Proposition 29. Theorem.

If a straight line fall on two parallel straight lines, then it shall make the alternate angles equal to one another, and the exterior angle equal to the interior opposite angle on the same side; and also the two interior angles on the same side equal to two right angles.



Let the straight line EF fall on the parallel straight lines AB, CD.

Then (i) *the angle* AGH *shall be equal to the alternate angle* GHD;

(ii) *the exterior angle* EGB *shall be equal to the interior opposite angle* GHD;

(iii) *the two interior angles* BGH, GHD *shall be together equal to two right angles.*

Proof. (i) For if the angle AGH be not equal to the angle GHD, one of them must be greater than the other.

If possible, let the angle AGH be greater than the angle GHD;

add to each the angle BGH:

then the angles AGH, BGH are together greater than the angles BGH, GHD.

But the adjacent angles AGH, BGH are together equal to two right angles; I. 13.

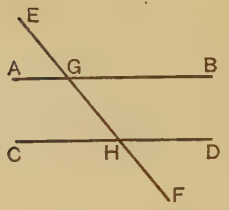
therefore the angles BGH, GHD are together less than two right angles;

therefore, *by Axiom 12*, AB and CD meet towards B and D.

But they never meet, since they are parallel. *Hyp.*

Therefore the angle AGH is not unequal to the angle GHD: that is, the angle AGH is equal to the alternate angle GHD.

(*Over*)



(ii) Again, because the angle AGH is equal to the vertically opposite angle EGB; I. 15.
and because the angle AGH is equal to the angle GHD;
Proved.
therefore the exterior angle EGB is equal to the interior opposite angle GHD.

(iii) Lastly, the angle EGB is equal to the angle GHD;
Proved.
add to each the angle BGH;
then the angles EGB, BGH are together equal to the angles BGH, GHD.
But the adjacent angles EGB, BGH are together equal to two right angles: I. 13.
therefore also the two interior angles BGH, GHD are together equal to two right angles. Q.E.D.

EXERCISES ON PROPOSITIONS 27, 28, 29.

1. Two straight lines AB, CD bisect one another at O: shew that the straight lines joining AC and BD are parallel. [I. 27.]

2. *Straight lines which are perpendicular to the same straight line are parallel to one another.* [I. 27 or I. 28.]

3. *If a straight line meet two or more parallel straight lines, and is perpendicular to one of them, it is also perpendicular to all the others.* [I. 29.]

4. *If two straight lines are parallel to two other straight lines, each to each, then the angles contained by the first pair are equal respectively to the angles contained by the second pair.* [I. 29.]

Note on the Twelfth Axiom.

Euclid's twelfth Axiom is unsatisfactory as the basis of a theory of parallel straight lines. It cannot be regarded as either simple or self-evident, and it therefore falls short of the essential characteristics of an axiom: nor is the difficulty entirely removed by considering it as a corollary to Proposition 17, of which it is the converse.

Of the many substitutes which have been proposed, we need only notice the following:

Axiom. *Two intersecting straight lines cannot be both parallel to a third straight line.*

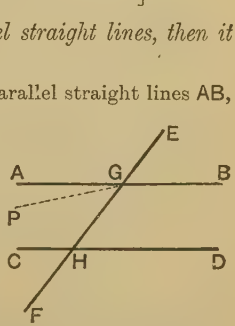
This statement is known as **Playfair's Axiom**; and though it is not altogether free from objection, it is no doubt simpler and more fundamental than that employed by Euclid, and more readily admitted without proof.

Propositions 27 and 28 having been proved in the usual way, the first part of Proposition 29 is then given thus.

Proposition 29. [Alternative Proof.]

If a straight line fall on two parallel straight lines, then it shall make the alternate angles equal.

Let the straight line EF meet the two parallel straight lines AB, CD at G and H.



Then shall the alternate angles AGH, GHD *be equal.*

For if the angle AGH is not equal to the angle GHD:

at G in the straight line HG make the angle HGP equal to the angle GHD, and alternate to it. I. 23.

Then PG and CD are parallel. I. 27.

But AB and CD are parallel: *Hyp.*

therefore the two intersecting straight lines AG, PG are both parallel to CD:

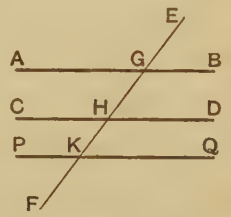
which is impossible. *Playfair's Axiom.*

Therefore the angle AGH is not unequal to the angle GHD; that is, the alternate angles AGH, GHD are equal. Q.E.D.

The second and third parts of the Proposition may then be deduced as in the text; and Euclid's Axiom 12 follows as a Corollary.

Proposition 30. Theorem.

Straight lines which are parallel to the same straight line are parallel to one another.



Let the straight lines AB, CD be each parallel to the straight line PQ.

Then shall AB *and* CD *be parallel to one another.*

Construction. Draw any straight line EF cutting AB, CD, and PQ in the points G, H, and K.

Proof. Then because AB and PQ are parallel, and EF meets them,
therefore the angle AGK is equal to the alternate angle GKQ I. 29.

And because CD and PQ are parallel, and EF meets them,
therefore the exterior angle GHD is equal to the interior opposite angle GKQ. I. 29.

Therefore the angle AGH is equal to the angle GHD;
and these are alternate angles;
therefore AB and CD are parallel. I. 27.

Q.E.D.

Note. If PQ lies between AB and CD, the Proposition may be established in a similar manner, though in this case it scarcely needs proof; for it is inconceivable that two straight lines, which do not meet an intermediate straight line, should meet one another.

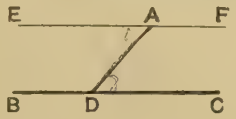
The truth of this Proposition may be readily deduced from Playfair's Axiom, of which it is the converse.

For if AB and CD were not parallel, they would meet when produced. Then there would be two intersecting straight lines both parallel to a third straight line: which is impossible.

Therefore AB and CD never meet; that is, they are parallel.

PROPOSITION 31. PROBLEM.

To draw a straight line through a given point parallel to a given straight line.



Let A be the given point, and BC the given straight line.

It is required to draw through A *a straight line parallel to* BC.

Construction. In BC take any point D; and join AD.

At the point A in DA, make the angle DAE equal to the angle ADC, and alternate to it, I. 23.

and produce EA to F.

Then shall EF *be parallel to* BC.

Proof. Because the straight line AD, meeting the two straight lines EF, BC, makes the alternate angles EAD, ADC equal; *Constr.*

therefore EF is parallel to BC; I. 27.

and it has been drawn through the given point A.

Q.E.F.

EXERCISES.

1. Any straight line drawn parallel to the base of an isosceles triangle makes equal angles with the sides.

2. If from any point in the bisector of an angle a straight line is drawn parallel to either arm of the angle, the triangle thus formed is isosceles.

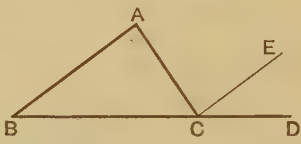
3. From a given point draw a straight line that shall make with a given straight line an angle equal to a given angle.

4. From X, a point in the base BC of an isosceles triangle ABC, a straight line is drawn at right angles to the base, cutting AB in Y, and CA produced in Z: shew the triangle AYZ is isosceles.

5. If the straight line which bisects an exterior angle of a triangle is parallel to the opposite side, shew that the triangle is isosceles.

PROPOSITION 32. THEOREM.

If a side of a triangle be produced, then the exterior angle shall be equal to the sum of the two interior opposite angles, also the three interior angles of a triangle are together equal to two right angles.



Let ABC be a triangle, and let one of its sides BC be produced to D.

Then (i) *the exterior angle* ACD *shall be equal to the sum of the two interior opposite angles* CAB, ABC;

(ii) *the three interior angles* ABC, BCA, CAB *shall be together equal to two right angles.*

Construction. Through C draw CE parallel to BA. I. 31.

Proof. (i) Then because BA and CE are parallel, and AC meets them,

therefore the angle ACE is equal to the alternate angle CAB. I. 29.

Again, because BA and CE are parallel, and BD meets them,

therefore the exterior angle ECD is equal to the interior opposite angle ABC. I. 29.

Therefore the whole exterior angle ACD is equal to the sum of the two interior opposite angles CAB, ABC.

(ii) Again, since the angle ACD is equal to the sum of the angles CAB, ABC; *Proved.*

to each of these equals add the angle BCA:

then the angles BCA, ACD are together equal to the three angles BCA, CAB, ABC.

But the adjacent angles BCA, ACD are together equal to two right angles. I. 13.

Therefore also the angles BCA, CAB, ABC are together equal to two right angles. Q.E.D.

From this Proposition we draw the following important inferences.

1. *If two triangles have two angles of the one equal to two angles of the other, each to each, then the third angle of the one is equal to the third angle of the other.*

2. *In any right-angled triangle the two acute angles are complementary.*

3. *In a right-angled isosceles triangle each of the equal angles is half a right angle.*

4. *If one angle of a triangle is equal to the sum of the other two, the triangle is right-angled.*

5. *The sum of the angles of any quadrilateral figure is equal to four right angles.*

6. *Each angle of an equilateral triangle is two-thirds of a right angle.*

EXERCISES ON PROPOSITION 32.

1. Prove that the three angles of a triangle are together equal to two right angles,

   (i) by drawing through the vertex a straight line parallel to the base;

   (ii) by joining the vertex to any point in the base.

2. If the base of any triangle is produced both ways, shew that the sum of the two exterior angles diminished by the vertical angle is equal to two right angles.

3. *If two straight lines are perpendicular to two other straight lines, each to each, the acute angle between the first pair is equal to the acute angle between the second pair.*

4. *Every right-angled triangle is divided into two isosceles triangles by a straight line drawn from the right angle to the middle point of the hypotenuse.*

Hence the joining line is equal to half the hypotenuse.

5. *Draw a straight line at right angles to a given finite straight line from one of its extremities, without producing the given straight line.*

[Let AB be the given straight line. On AB describe any isosceles triangle ACB. Produce BC to D, making CD equal to BC. Join AD. Then shall AD be perpendicular to AB.]

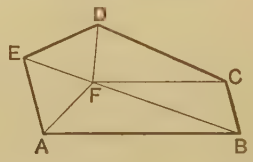
6. *Trisect a right angle.*

7. The angle contained by the bisectors of the angles at the base of an isosceles triangle is equal to an exterior angle formed by producing the base.

8. The angle contained by the bisectors of two adjacent angles of a quadrilateral is equal to half the sum of the remaining angles.

The following theorems were added as corollaries to Proposition 32 by Robert Simson, who edited Euclid's text in 1756.

COROLLARY 1. *All the interior angles of any rectilineal figure, together with four right angles, are equal to twice as many right angles as the figure has sides.*



Let ABCDE be any rectilineal figure.
Take F, any point within it,
and join F to each of the angular points of the figure.

Then the figure is divided into as many triangles as it has sides.

And the three angles of each triangle are together equal to two right angles. I. 32.

Hence *all* the angles of *all* the triangles are together equal to twice as many right angles as the figure has sides.

But all the angles of all the triangles make up all the interior angles of the figure, together with the angles at F, which are equal to four right angles. I. 15, *Cor.*

Therefore all the interior angles of the figure, together with four right angles, are equal to twice as many right angles as the figure has sides. Q.E.D.

COROLLARY 2. *If the sides of a rectilineal figure, which has no re-entrant angle, are produced in order, then all the exterior angles so formed are together equal to four right angles.*

For at each angular point of the figure, the interior angle and the exterior angle are together equal to two right angles. I. 13.

Therefore all the interior angles, with all the exterior angles, are together equal to twice as many right angles as the figure has sides.

But all the interior angles, with four right angles, are together equal to twice as many right angles as the figure has sides. I. 32, *Cor.* 1.

Therefore all the interior angles, with all the exterior angles, are together equal to all the interior angles, with four right angles.

Therefore the exterior angles are together equal to four right angles. Q.E.D.

EXERCISES ON SIMSON'S COROLLARIES.

[A polygon is said to be **regular** when it has all its sides and all its angles equal.]

1. Express in terms of a right angle the magnitude of each angle of (i) a regular hexagon, (ii) a regular octagon.

2. If one side of a regular hexagon is produced, shew that the exterior angle is equal to the angle of an equilateral triangle.

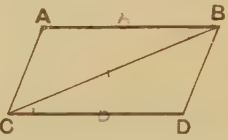
3. Prove Simson's first Corollary by joining one vertex of the rectilineal figure to each of the other vertices.

4. Find the magnitude of each angle of a regular polygon of n sides.

5. If the alternate sides of any polygon be produced to meet, the sum of the included angles, together with eight right angles, will be equal to twice as many right angles as the figure has sides.

Proposition 33. Theorem.

The straight lines which join the extremities of two equal and parallel straight lines towards the same parts are themselves equal and parallel.



Let AB and CD be equal and parallel straight lines; and let them be joined towards the same parts by the straight lines AC and BD.

Then shall AC *and* BD *be equal and parallel.*

Construction. Join BC.

Proof. Then because AB and CD are parallel, and BC meets them,
therefore the angle ABC is equal to the alternate angle DCB. I. 29.

Now in the triangles ABC, DCB,

Because { AB is equal to DC, *Hyp.*
and BC is common to both;
also the angle ABC is equal to the angle DCB; *Proved.*

therefore the triangle ABC is equal to the triangle DCB in all respects; I. 4.

so that the base AC is equal to the base DB,
and the angle ACB equal to the angle DBC.

But these are alternate angles.

Therefore AC and BD are parallel: I. 27.

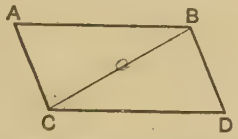
and it has been shewn that they are also equal.

Q.E.D.

Definition. A **Parallelogram** is a four-sided figure whose opposite sides are parallel.

PROPOSITION 34. THEOREM.

The opposite sides and angles of a parallelogram are equal to one another, and each diagonal bisects the parallelogram.



Let ACDB be a parallelogram, of which BC is a diagonal.

Then shall the opposite sides and angles of the figure be equal to one another; and the diagonal BC *shall bisect it*

PROOF. Because AB and CD are parallel, and BC meets them,

therefore the angle ABC is equal to the alternate angle DCB; I. 29.

Again, because AC and BD are parallel, and BC meets them,

therefore the angle ACB is equal to the alternate angle DBC. I. 29.

Hence in the triangles ABC, DCB,

Because { the angle ABC is equal to the angle DCB,
and the angle ACB is equal to the angle DBC;
also the side BC is common to both;

therefore the triangle ABC is equal to the triangle DCB in all respects; I. 26.

so that AB is equal to DC, and AC to DB;
and the angle BAC is equal to the angle CDB.

Also, because the angle ABC is equal to the angle DCB,
and the angle CBD equal to the angle BCA,

therefore the whole angle ABD is equal to the whole angle DCA.

And the triangles ABC, DCB having been proved equal in all respects are equal in area.

Therefore the diagonal BC bisects the parallelogram ACDB.

Q.E.D.

EXERCISES ON PARALLELOGRAMS.

1. *If one angle of a parallelogram is a right angle, all its angles are right angles.*

2. *If the opposite sides of a quadrilateral are equal, the figure is a parallelogram.*

3. *If the opposite angles of a quadrilateral are equal, the figure is a parallelogram.*

4. *If a quadrilateral has all its sides equal and one angle a right angle, all its angles are right angles.*

5. *The diagonals of a parallelogram bisect each other.*

6. *If the diagonals of a quadrilateral bisect each other, the figure is a parallelogram.*

7. If two opposite angles of a parallelogram are bisected by the diagonal which joins them, the figure is equilateral.

8. If the diagonals of a parallelogram are equal, all its angles are right angles.

9. In a parallelogram which is not rectangular the diagonals are unequal.

10. Any straight line drawn through the middle point of a diagonal of a parallelogram and terminated by a pair of opposite sides, is bisected at that point.

11. *If two parallelograms have two adjacent sides of one equal to two adjacent sides of the other, each to each, and one angle of one equal to one angle of the other, the parallelograms are equal in all respects.*

12. *Two rectangles are equal if two adjacent sides of one are equal to two adjacent sides of the other, each to each.*

13. In a parallelogram the perpendiculars drawn from one pair of opposite angles to the diagonal which joins the other pair are equal.

14. If ABCD is a parallelogram, and X, Y respectively the middle points of the sides AD, BC ; shew that the figure AYCX is a parallelogram.

MISCELLANEOUS EXERCISES ON SECTIONS I. AND II.

1. Shew that the construction in Proposition 2 may generally be performed in eight different ways. Point out the exceptional case.

2. The bisectors of two vertically opposite angles are in the same straight line.

3. In the figure of Proposition 16, if AF is joined, shew

(i) that AF is equal to BC;

(ii) that the triangle ABC is equal to the triangle CFA in all respects.

4. ABC is a triangle right-angled at B, and BC is produced to D: shew that the angle ACD is obtuse.

5. Shew that in any regular polygon of n sides each angle contains $\frac{2(n-2)}{n}$ right angles.

6. The angle contained by the bisectors of the angles at the base of any triangle is equal to the vertical angle together with half the sum of the base angles.

7. The angle contained by the bisectors of two exterior angles of any triangle is equal to half the sum of the two corresponding interior angles.

8. If perpendiculars are drawn to two intersecting straight lines from any point between them, shew that the bisector of the angle between the perpendiculars is parallel to (or coincident with) the bisector of the angle between the given straight lines.

9. If two points P, Q be taken in the equal sides of an isosceles triangle ABC, so that BP is equal to CQ, shew that PQ is parallel to BC.

10. ABC and DEF are two triangles, such that AB, BC are equal and parallel to DE, EF, each to each; shew that AC is equal and parallel to DF.

11. Prove the second Corollary to Prop. 32 by drawing through any angular point lines parallel to all the sides.

12. If two sides of a quadrilateral are parallel, and the remaining two sides equal but not parallel, shew that the opposite angles are supplementary: also that the diagonals are equal.

SECTION III.

THE AREAS OF PARALLELOGRAMS AND TRIANGLES.

Hitherto when two figures have been said to be *equal*, it has been implied that they are *identically* equal, that is, equal in all respects.

But figures may be equal *in area* without being equal in all respects, that is, without having the same shape.

The present section deals with parallelograms and triangles which are equal in area but not necessarily identically equal.

[The ultimate test of equality, as we have already seen, is afforded by Axiom 8, which asserts that magnitudes which *may be made to coincide with one another* are equal. Now figures which are not equal in all respects, cannot be made to coincide without first undergoing some change of form : hence the method of direct *superposition* is unsuited to the purposes of the present section.

We shall see however from Euclid's proof of Proposition 35, that two figures which are not identically equal, may nevertheless be so related to a third figure, that it is possible to infer the equality of their areas.]

DEFINITIONS.

1. The **Altitude** of a parallelogram with reference to a given side as base, is the perpendicular distance between the base and the opposite side.

2. The **Altitude** of a triangle with reference to a given side as base, is the perpendicular distance of the opposite vertex from the base.

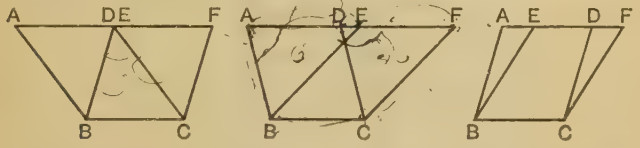
[From this point the following symbols will be introduced into the text :

$=$ for *is equal to* ; $\therefore$ for *therefore*.

If it is thought desirable to shorten *written work* by the use of symbols and abbreviations, it is strongly recommended that only some well recognized system should be allowed, such, for example, as that given on page 11.]

PROPOSITION 35. THEOREM.

Parallelograms on the same base, and between the same parallels, are equal in area.



Let the parallelograms ABCD, EBCF be on the same base BC, and between the same parallels BC, AF.

Then shall the parallelogram ABCD *be equal in area to the parallelogram* EBCF.

CASE I. If the sides AD, EF, opposite to the base BC, are terminated at the same point D:

then each of the parallelograms ABCD, EBCF is double of the triangle BDC; I. 34.

∴ they are equal to one another. *Ax.* 6.

CASE II. But if the sides AD, EF are not terminated at the same point:

then because ABCD is a parallelogram,

∴ the side AD = the opposite side BC; I. 34.

similarly EF = BC;

∴ AD = EF. *Ax.* 1.

∴ the whole, or remainder, EA = the whole, or remainder, FD.

Then in the triangles FDC, EAB,

Because { FD = EA, *Proved.*
and the side DC = the opposite side AB, I. 34.
also the exterior angle FDC = the interior opposite angle EAB, I. 29.

∴ the triangle FDC = the triangle EAB. I. 4.

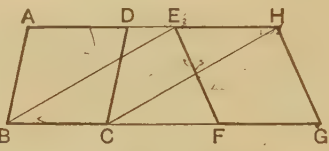
From the whole figure ABCF take the triangle FDC; and from the same figure take the equal triangle EAB;

then the remainders are equal. *Ax.* 3.

Therefore the parallelogram ABCD is equal to the parallelogram EBCF.

Proposition 36. Theorem.

Parallelograms on equal bases, and between the same parallels, are equal in area.



Let ABCD, EFGH be parallelograms on equal bases BC, FG, and between the same parallels AH, BG.

Then shall the parallelogram ABCD *be equal to the parallelogram* EFGH.

Construction. Join BE, CH.

Proof. Then because BC = FG; *Hyp.*
and the side FG = the opposite side EH; I. 34.
∴ BC = EH: *Ax.* 1.
and BC is parallel to EH; *Hyp.*
∴ BE and CH are also equal and parallel. I. 33.
Therefore EBCH is a parallelogram. *Def.* 36.

Now the parallelograms ABCD, EBCH are on the same base BC, and between the same parallels BC, AH;
∴ the parallelogram ABCD = the parallelogram EBCH. I. 35.

Also the parallelograms EFGH, EBCH are on the same base EH, and between the same parallels EH, BG;
∴ the parallelogram EFGH = the parallelogram EBCH. I. 35.

Therefore the parallelogram ABCD is equal to the parallelogram EFGH. *Ax.* 1.

Q.E.D.

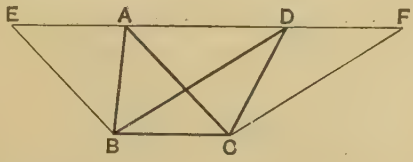
From the last two Propositions we infer that:

(i) *A parallelogram is equal in area to a rectangle of equal base and equal altitude.*

(ii) *Parallelograms on equal bases and of equal altitudes are equal in area.*

PROPOSITION 37. THEOREM.

Triangles on the same base, and between the same parallels, are equal in area.



Let the triangles ABC, DBC be upon the same base BC, and between the same parallels BC, AD.

Then shall the triangle ABC *be equal to the triangle* DBC.

Construction. Through B draw BE parallel to CA, to meet DA produced in E; I. 31.
through C draw CF parallel to BD, to meet AD produced in F.

Proof. Then, by construction, each of the figures EBCA, DBCF is a parallelogram. *Def.* 36.

And since they are on the same base BC, and between the same parallels BC, EF;

∴ the parallelogram EBCA = the parallelogram DBCF. I. 35.

Now the diagonal AB bisects EBCA; I. 34.

∴ the triangle ABC is half the parallelogram EBCA.

And the diagonal DC bisects DBCF; I. 34.

∴ the triangle DBC is half the parallelogram DBCF.

And the halves of equal things are equal. *Ax.* 7.

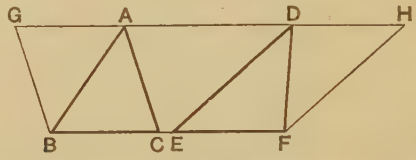
Therefore the triangle ABC is equal to the triangle DBC.

Q.E.D.

[For Exercises see page 79.]

Proposition 38. Theorem.

Triangles on equal bases, and between the same parallels, are equal in area.



Let the triangles ABC, DEF be on equal bases BC, EF, and between the same parallels BF, AD.

Then shall the triangle ABC *be equal to the triangle* DEF.

Construction. Through B draw BG parallel to CA, to meet DA produced in G; I. 31.
through F draw FH parallel to ED, to meet AD produced in H.

Proof. Then, by construction, each of the figures GBCA, DEFH is a parallelogram. *Def.* 36.

And since they are on equal bases BC, EF, and between the same parallels BF, GH;

∴ the parallelogram GBCA = the parallelogram DEFH. I. 36.

Now the diagonal AB bisects GBCA; I. 34.

∴ the triangle ABC is half the parallelogram GBCA.

And the diagonal DF bisects DEFH; I. 34.

∴ the triangle DEF is half the parallelogram DEFH.

And the halves of equal things are equal. *Ax.* 7.

Therefore the triangle ABC is equal to the triangle DEF.

Q.E.D.

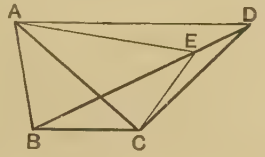
From this Proposition we infer that:

(i) *Triangles on equal bases and of equal altitude are equal in area.*

(ii) *Of two triangles of the same altitude, that is the greater which has the greater base; and of two triangles on the same base, or on equal bases, that is the greater which has the greater altitude.*

PROPOSITION 39. THEOREM.

Equal triangles on the same base, and on the same side of it, are between the same parallels.



Let the triangles ABC, DBC which stand on the same base BC, and on the same side of it be equal in area.

Then shall the triangles ABC, DBC *be between the same parallels; that is, if* AD *be joined,* AD *shall be parallel to* BC.

Construction. For if AD be not parallel to BC,
if possible, through A draw AE parallel to BC, I. 31.
meeting BD, or BD produced, in E.
Join EC.

Proof. Now the triangles ABC, EBC are on the same base BC, and between the same parallels BC, AE;
∴ the triangle ABC = the triangle EBC. I. 37.
But the triangle ABC = the triangle DBC; *Hyp.*
∴ the triangle DBC = the triangle EBC;
that is, the whole is equal to a part; which is impossible.
∴ AE is not parallel to BC.

Similarly it can be shewn that no other straight line through A, except AD, is parallel to BC.

Therefore AD is parallel to BC.

Q.E.D.

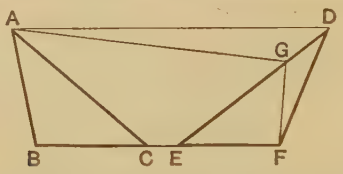
From this Proposition it follows that:

Equal triangles on the same base have equal altitudes.

[For Exercises see page 79.]

Proposition 40. Theorem.

Equal triangles, on equal bases in the same straight line, and on the same side of it, are between the same parallels.



Let the triangles ABC, DEF which stand on equal bases BC, EF, in the same straight line BF, and on the same side of it, be equal in area.

Then shall the triangles ABC, DEF *be between the same parallels; that is, if* AD *be joined,* AD *shall be parallel to* BF.

Construction. For if AD be not parallel to BF,
if possible, through A draw AG parallel to BF, I. 31.
meeting ED, or ED produced, in G.
Join GF.

Proof. Now the triangles ABC, GEF are on equal bases BC, EF, and between the same parallels BF, AG;
∴ the triangle ABC = the triangle GEF. I. 38.
But the triangle ABC = the triangle DEF: *Hyp.*
∴ the triangle DEF = the triangle GEF:
that is, the whole is equal to a part; which is impossible.
∴ AG is not parallel to BF.

Similarly it can be shewn that no other straight line through A, except AD, is parallel to BF.

Therefore AD is parallel to BF.

Q.E.D.

From this Proposition it follows that:

(i) *Equal triangles on equal bases have equal altitudes.*

(ii) *Equal triangles of equal altitudes have equal bases.*

EXERCISES ON PROPOSITIONS 37–40.

DEFINITION. Each of the three straight lines which join the angular points of a triangle to the middle points of the opposite sides is called a **Median** of the triangle.

ON PROP. 37.

1. If, in the figure of Prop. 37, AC and BD intersect in K, shew that
   (i) the triangles AKB, DKC are equal in area.
   (ii) the quadrilaterals EBKA, FCKD are equal.

2. In the figure of I. 16, shew that the triangles ABC, FBC are equal in area.

3. On the base of a given triangle construct a second triangle, equal in area to the first, and having its vertex in a given straight line.

4. Describe an isosceles triangle equal in area to a given triangle and standing on the same base.

ON PROP. 38.

5. *A triangle is divided by each of its medians into two parts of equal area.*

6. A parallelogram is divided by its diagonals into four triangles of equal area.

7. ABC is a triangle, and its base BC is bisected at X; if Y be any point in the median AX, shew that the triangles ABY, ACY are equal in area.

8. In AC, a diagonal of the parallelogram ABCD, any point X is taken, and XB, XD are drawn: shew that the triangle BAX is equal to the triangle DAX.

9. If two triangles have two sides of one respectively equal to two sides of the other, and the angles contained by those sides *supplementary*, the triangles are equal in area.

ON PROP. 39.

10. *The straight line which joins the middle points of two sides of a triangle is parallel to the third side.*

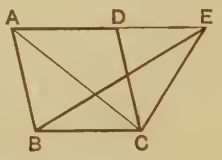
11. *If two straight lines* AB, CD *intersect in* O, *so that the triangle* AOC *is equal to the triangle* DOB, *shew that* AD *and* CB *are parallel.*

ON PROP. 40.

12. Deduce Prop. 40 from Prop. 39 by joining AE, AF in the figure of page 78.

PROPOSITION 41. THEOREM.

If a parallelogram and a triangle be on the same base and between the same parallels, the parallelogram shall be double of the triangle.



Let the parallelogram ABCD, and the triangle EBC be upon the same base BC, and between the same parallels BC, AE.

Then shall the parallelogram ABCD *be double of the triangle* EBC.

Construction. Join AC.

Proof. Now the triangles ABC, EBC are on the same base BC, and between the same parallels BC, AE;

$\therefore$ the triangle ABC = the triangle EBC. I. 37.

And since the diagonal AC bisects ABCD; I. 34.

$\therefore$ the parallelogram ABCD is double of the triangle ABC.

Therefore the parallelogram ABCD is also double of the triangle EBC. Q.E.D.

EXERCISES.

1. ABCD is a parallelogram, and X, Y are the middle points of the sides AD, BC; if Z is any point in XY, or XY produced, shew that the triangle AZB is one quarter of the parallelogram ABCD.

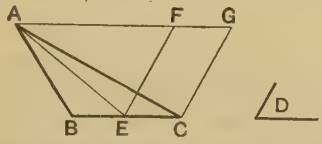
2. Describe a right-angled isosceles triangle equal to a given square.

3. If ABCD is a parallelogram, and X, Y any points in DC and AD respectively: shew that the triangles AXB, BYC are equal in area.

4. ABCD is a parallelogram, and P is any point within it; shew that the sum of the triangles PAB, PCD is equal to half the parallelogram.

PROPOSITION 42. PROBLEM.

To describe a parallelogram that shall be equal to a given triangle, and have one of its angles equal to a given angle.



Let ABC be the given triangle, and D the given angle.

It is required to describe a parallelogram equal to ABC, *and having one of its angles equal to* D.

Construction. Bisect BC at E. I. 10.

At E in CE, make the angle CEF equal to D; I. 23.

through A draw AFG parallel to EC; I. 31.

and through C draw CG parallel to EF.

Then FECG *shall be the parallelogram required.*

Join AE.

Proof. Now the triangles ABE, AEC are on equal bases BE, EC, and between the same parallels;

$\therefore$ the triangle ABE = the triangle AEC; I. 38.

$\therefore$ the triangle ABC is double of the triangle AEC.

But FECG is a parallelogram by construction; *Def.* 36

and it is double of the triangle AEC,

being on the same base EC, and between the same parallels EC and AG. I. 41.

Therefore the parallelogram FECG is equal to the triangle ABC;

and it has one of its angles CEF equal to the given angle D.

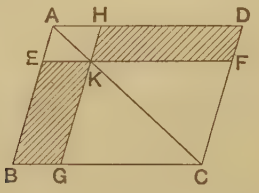
Q.E.F.

EXERCISES.

1. Describe a parallelogram equal to a given square standing on the same base, and having an angle equal to half a right angle.

2. Describe a rhombus equal to a given parallelogram and standing on the same base. When does the construction fail?

DEFINITION. If in the diagonal of a parallelogram any point is taken, and straight lines are drawn through it parallel to the sides of the parallelogram; then of the four parallelograms into which the whole figure is divided, the two through which the diagonal passes are called **Parallelograms about that diagonal**, and the other two, which with these make up the whole figure, are called the **complements** of the parallelograms about the diagonal.

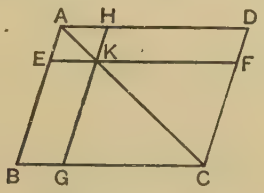


Thus in the figure given above, AEKH, KGCF are parallelograms about the diagonal AC; and the shaded figures HKFD, EBGK are the complements of those parallelograms.

NOTE. A parallelogram is often named by *two* letters only, these being placed at opposite angular points.

PROPOSITION 43. THEOREM.

The complements of the parallelograms about the diagonal of any parallelogram, are equal to one another.



Let ABCD be a parallelogram, and KD, KB the complements of the parallelograms EH, GF about the diagonal AC.

Then shall the complement BK *be equal to the complement* KD.

Proof. Because EH is a parallelogram, and AK its diagonal,
∴ the triangle AEK = the triangle AHK. I. 34.

Similarly the triangle KGC = the triangle KFC.

Hence the triangles AEK, KGC are together equal to the triangles AHK, KFC.

But since the diagonal AC bisects the parallelogram ABCD;
∴ the whole triangle ABC = the whole triangle ADC. I. 34.

Therefore the remainder, the complement BK, is equal to the remainder, the complement KD. Q.E.D.

EXERCISES.

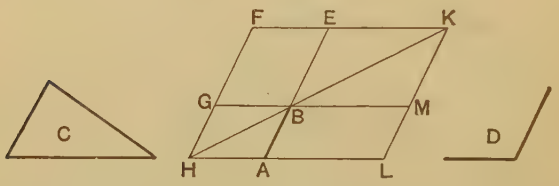
In the figure of Prop. 43, prove that

(i) The parallelogram ED is equal to the parallelogram BH.

(ii) If KB, KD are joined, the triangle AKB is equal to the triangle AKD.

Proposition 44. Problem.

To a given straight line to apply a parallelogram which shall be equal to a given triangle, and have one of its angles equal to a given angle.



Let AB be the given straight line, C the given triangle, and D the given angle.

It is required to apply to the straight line AB *a parallelogram equal to the triangle* C, *and having an angle equal to the angle* D.

Construction. On AB produced describe a parallelogram BEFG equal to the triangle C, and having the angle EBG equal to the angle D. I. 22 *and* I. 42*.

Through A draw AH parallel to BG or EF, to meet FG produced in H. I. 31.

Join HB.

Then because AH and EF are parallel, and HF meets them,

∴ the angles AHF, HFE together = two right angles. I. 29.

Hence the angles BHF, HFE are together less than two right angles;

∴ HB and FE will meet if produced towards B and E. *Ax.* 12.

Produce HB and FE to meet at K.

Through K draw KL parallel to EA or FH; I. 31. and produce HA, GB to meet KL in the points L and M.

Then shall BL *be the parallelogram required.*

Proof. Now FHLK is a parallelogram, *Constr.*
and LB, BF are the complements of the parallelograms about the diagonal HK:

$\therefore$ the complement LB = the complement BF. I. 43.

But the triangle C = the figure BF; *Constr.*
$\therefore$ the figure LB = the triangle C.

Again the angle ABM = the vertically opposite angle GBE;
also the angle D = the angle GBE; *Constr.*
$\therefore$ the angle ABM = the angle D.

Therefore the parallelogram LB, which is applied to the straight line AB, is equal to the triangle C, and has the angle ABM equal to the angle D. Q.E.F.

*This step of the construction is effected by first describing on AB produced a triangle whose sides are respectively equal to those of the triangle C (I. 22); and by then making a parallelogram equal to the triangle so drawn, and having an angle equal to D (I. 42).

QUESTIONS FOR REVISION.

1. Quote Euclid's Twelfth Axiom. What objections have been raised to it, and what substitute for it has been suggested?

2. Which of Euclid's Propositions, dealing with parallel straight lines, depends on Axiom 12? Furnish an alternative proof.

3. *Straight lines which are parallel to the same straight line are parallel to one another* [Prop. 30]. Deduce this from Playfair's Axiom.

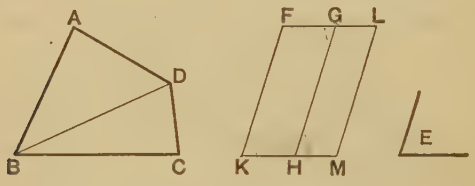
4. Define *a parallelogram*, *an altitude* of a triangle, *a median* of a triangle, *parallelograms about the diagonal of a parallelogram.*

5. What is meant by *superposition*? On what Axiom does this method depend? Give instances of figures which are equal *in area*, but which cannot be superposed.

6. In fig. 2 of Prop. 35 shew how one parallelogram may be cut into pieces, which, when fitted together in other positions, make up the other parallelogram.

PROPOSITION 45. PROBLEM.

To describe a parallelogram equal to a given rectilineal figure, and having an angle equal to a given angle.



Let ABCD be the given rectilineal figure, and E the given angle.

It is required to describe a parallelogram equal to ABCD, *and having an angle equal to* E.

Suppose the given rectilineal figure to be a quadrilateral.

Construction. Join BD.

Describe the parallelogram FH equal to the triangle ABD, and having the angle FKH equal to the angle E. I. 42.

To GH apply the parallelogram GM, equal to the triangle DBC, and having the angle GHM equal to E. I. 44.

Then shall FKML *be the parallelogram required.*

Proof. Because each of the angles GHM, FKH = the angle E;
∴ the angle FKH = the angle GHM.

To each of these equals add the angle GHK;
then the angles FKH, GHK together = the angles GHM, GHK.

But since FK, GH are parallel, and KH meets them;
∴ the angles FKH, GHK together = two right angles; I. 29.
∴ also the angles GHM, GHK together = two right angles;
∴ KH, HM are in the same straight line. I. 14.

Again, because KM, FG are parallel, and HG meets them,
∴ the angle MHG = the alternate angle HGF. I. 29.

To each of these equals add the angle HGL;
then the angles MHG, HGL together = the angles HGF, HGL.

But because HM, GL are parallel, and HG meets them,
∴ the angles MHG, HGL together = two right angles: I. 29.
∴ also the angles HGF, HGL together = two right angles:
∴ FG, GL are in the same straight line. I. 14.

And because KF and ML are each parallel to HG, *Constr.*
therefore KF is parallel to ML; I. 30.
and KM, FL are parallel; *Constr.*
∴ FKML is a parallelogram. *Def.* 36.

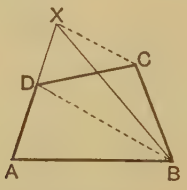
Again, because the parallelogram FH = the triangle ABD,
and the parallelogram GM = the triangle DBC; *Constr.*
∴ the whole parallelogram FKML = the whole figure ABCD;
and it has the angle FKM equal to the angle E.

By a series of similar steps, a parallelogram may be constructed equal to a rectilineal figure of more than four sides. Q.E.F.

The following Problem is important, and furnishes a useful application of the principles of the foregoing propositions.

ADDITIONAL PROBLEM.

To describe a triangle equal in area to a given quadrilateral.



Let ABCD be the given quadrilateral.

It is required to describe a triangle equal to ABCD *in area.*

Construction. Join BD.

Through C draw CX parallel to BD, meeting AD produced in X.

Join BX.

Then XAB shall be the required triangle.

Proof. Now the triangles XDB, CDB are on the same base DB and between the same parallels DB, XC;

∴ the triangle XDB = the triangle CDB in area. I. 37.

To each of these equals add the triangle ADB;

then the triangle XAB = the figure ABCD.

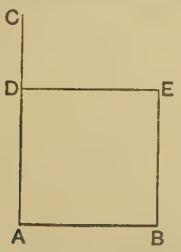
EXERCISE.

Construct a rectilineal figure equal to a given rectilineal figure, and having fewer sides by one than the given figure.

Hence shew how to construct a triangle equal to a given rectilineal figure.

PROPOSITION 46. PROBLEM.

To describe a square on a given straight line.



Let AB be the given straight line.
It is required to describe a square on AB.

Constr. From A draw AC at right angles to AB ; I. 11.
and make AD equal to AB. I. 3.
Through D draw DE parallel to AB ; I. 31.
and through B draw BE parallel to AD, meeting DE in E.
Then shall ADEB *be a square.*

Proof. For, by construction, ADEB is a parallelogram :
$\therefore$ AB = DE, and AD = BE. I. 34.
But AD = AB ; *Constr.*
$\therefore$ the four straight lines AB, AD, DE, EB are all equal ;
that is, the figure ADEB is equilateral.

Again, since AB, DE are parallel, and AD meets them,
$\therefore$ the angles BAD, ADE together = two right angles ; I. 29.
but the angle BAD is a right angle ; *Constr.*
$\therefore$ also the angle ADE is a right angle.

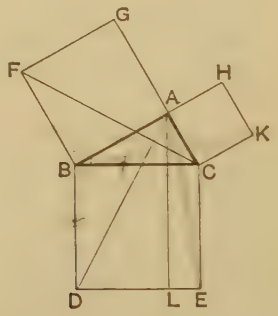
And the opposite angles of a parallelogram are equal ; I. 34.
$\therefore$ each of the angles DEB, EBA is a right angle :
that is the figure ADEB is rectangular.
Hence it is a square, and it is described on AB.

Q.E.F.

COROLLARY. *If one angle of a parallelogram is a right angle, all its angles are right angles.*

Proposition 47. Theorem.

In a right-angled triangle the square described on the hypotenuse is equal to the sum of the squares described on the other two sides.



Let ABC be a right-angled triangle, having the angle BAC a right angle.

Then shall the square described on the hypotenuse BC *be equal to the sum of the squares described on* BA, AC.

Construction. On BC describe the square BDEC : I. 46.
and on BA, AC describe the squares BAGF, ACKH.
Through A draw AL parallel to BD or CE ; I. 31.
and join AD, FC.

Proof. Then because each of the angles BAC, BAG is a right angle,
∴ CA and AG are in the same straight line. I. 14.

Now the angle CBD = the angle FBA,
for each of them is a right angle.
Add to each the angle ABC :
then the whole angle ABD = the whole angle FBC.

Then in the triangles ABD, FBC,

Because { AB = FB,
and BD = BC,
also the angle ABD = the angle FBC ; *Proved.*

∴ the triangle ABD = the triangle FBC. I. 4.

Now the parallelogram BL is double of the triangle ABD, being on the same base BD, and between the same parallels BD, AL. I. 41.

And the square GB is double of the triangle FBC, being on the same base FB, and between the same parallels FB, GC. I. 41.

But doubles of equals are equal : *Ax.* 6.

therefore the parallelogram BL = the square GB.

Similarly, by joining AE, BK it can be shewn that the parallelogram CL = the square CH.

Therefore the whole square BE = the sum of the squares GB, HC :

that is, the square described on the hypotenuse BC is equal to the sum of the squares described on the two sides BA, AC. Q.E.D.

NOTE. It is not necessary to the proof of this Proposition that the three squares should be described *external* to the triangle ABC ; and since *each* square may be drawn either *towards* or *away from* the triangle, it may be shewn that there are $2 \times 2 \times 2$, or *eight*, possible constructions.

Obs. The following properties of a square, though not formally enunciated by Euclid, are employed in subsequent proofs. [See I. 48.]

(i) *The squares on equal straight lines are equal.*

(ii) *Equal squares stand upon equal straight lines.*

EXERCISES ON PROPOSITION 47.

1. In the figure of this Proposition, shew that

   (i) If BG, CH are joined, these straight lines are parallel ;

   (ii) The points F, A, K are in one straight line ;

   (iii) FC and AD are at right angles to one another ;

   (iv) If GH, KE, FD are joined, the triangle GAH is equal to the given triangle in all respects ; and the triangles FBD, KCE are each equal in area to the triangle ABC.

[See Ex. 9, p. 79.]

2. On the sides AB, AC of *any* triangle ABC, squares ABFG, ACKH are described both toward the triangle, or both on the side remote from it : shew that the straight lines BH and CG are equal.

3. On the sides of any triangle ABC, equilateral triangles BCX, CAY, ABZ are described, all externally, or all towards the triangle : shew that AX, BY, CZ are all equal.

4. *The square described on the diagonal of a given square, is double of the given square.*

5. ABC *is an equilateral triangle, and* AX *is the perpendicular drawn from* A *to* BC *: shew that the square on* AX *is three times the square on* BX.

6. Describe a square equal to the sum of two given squares.

7. From the vertex A of a triangle ABC, AX is drawn perpendicular to the base : shew that the difference of the squares on the sides AB and AC, is equal to the difference of the squares on BX and CX, the segments of the base.

8. If from any point O within a triangle ABC, perpendiculars OX, OY, OZ are drawn to the sides BC, CA, AB respectively : shew that the sum of the squares on the segments AZ, BX, CY is equal to the sum of the squares on the segments AY, CX, BZ.

9. ABC is a triangle right-angled at A ; and the sides AB, AC are intersected by a straight line PQ, and BQ, PC are joined. Prove that the sum of the squares on BQ, PC is equal to the sum of the squares on BC, PQ.

10. In a right-angled triangle four times the sum of the squares on the two medians drawn from the acute angles is equal to five times the square on the hypotenuse.

NOTES ON PROPOSITION 47.

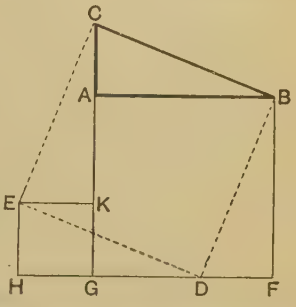
It is believed that Proposition 47 is due to Pythagoras, a Greek philosopher and mathematician, who lived about two centuries before Euclid.

Many experimental proofs of this theorem have been given by means of actual *dissection*: that is to say, it has been shewn how the squares on the sides containing the right angle may be cut up into pieces which, when fitted together in other positions, exactly make up the square on the hypotenuse. Two of these methods of dissection are given below.

I. In the adjoining diagram ABC is the given right-angled triangle, and the figures AF, HK are the squares on AB, AC, placed side by side.

FD is made equal to EH or AC;

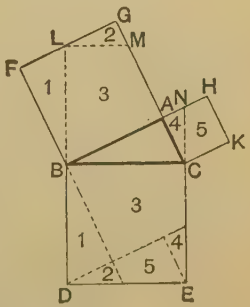
and the two squares AF, HK are cut along the lines ED, DB.



Then it will be found that the triangle EHD may be placed so as to fill up the space CAB; and the triangle BFD may be made to fill the space CKE.

Hence the two squares AF, HK may be fitted together so as to form the single figure CBDE, which will be found to be a perfect square, namely the square on the hypotenuse BC.

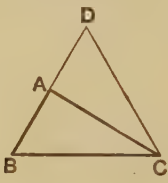
II. In the figure of I. 47, let DB and EC be produced to meet FG and AH in L and N respectively; and let LM be drawn parallel to BC.



Then it will be found that the several parts of the two squares FA, AK can be fitted together (in the places bearing corresponding numbers) so as exactly to fill up the square DC.

PROPOSITION 48. THEOREM.

If the square described on one side of a triangle be equal to the sum of the squares described on the other two sides, then the angle contained by these two sides shall be a right angle.



Let ABC be a triangle; and let the square described on BC be equal to the sum of the squares described on BA, AC.

Then shall the angle BAC *be a right angle.*

Construction. From A draw AD at right angles to AC; I. 11.
and make AD equal to AB. I. 3.
Join DC.

Proof. Then, because AD = AB, *Constr.*
∴ the square on AD = the square on AB.
To each of these add the square on CA;
then the sum of the squares on CA, AD = the sum of the squares on CA, AB.

But, because the angle DAC is a right angle, *Constr.*
∴ the square on DC = the sum of the squares on CA, AD. I. 47.
And, by hypothesis, the square on BC = the sum of the squares on CA, AB;
∴ the square on DC = the square on BC:
∴ also the side DC = the side BC.

Then in the triangles DAC, BAC,

Because { DA = BA, *Constr.*
and AC is common to both;
also the third side DC = the third side BC; *Proved.*

∴ the angle DAC = the angle BAC. I. 8.
But DAC is a right angle. *Constr.*
Therefore also BAC is a right angle. Q.E.D.

THEOREMS AND EXAMPLES ON BOOK I.

INTRODUCTORY.

HINTS TOWARDS THE SOLUTION OF GEOMETRICAL EXERCISES. ANALYSIS. SYNTHESIS.

It is commonly found that exercises in Pure Geometry present to a beginner far more difficulty than examples in any other branch of Elementary Mathematics. This seems to be due to the following causes :

(i) The variety of such exercises is practically unlimited ; and it is impossible to lay down for their treatment any definite methods, such for example as the rules of Elementary Arithmetic and Algebra.

(ii) The arrangement of Euclid's Propositions, though perhaps the most *convincing* of all forms of argument, affords in most cases little clue as to the way in which the proof or construction *was discovered.*

Euclid's propositions are arranged **synthetically** : that is to say, starting from the hypothesis or data, they first give a construction in accordance with postulates, and problems already solved ; then by successive steps based on known theorems, they prove what was required in the enunciation.

Thus Geometrical Synthesis is a *building up* of *known* results, in order to obtain a *new* result.

But as this is not the way in which constructions or proofs are usually discovered, we draw the student's attention to the following hints.

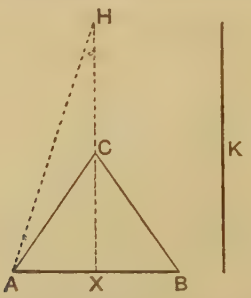
Begin by *assuming* the result it is desired to establish ; then by working backwards, trace the consequences of the assumption, and try to ascertain its dependence on some simpler theorem which is already known to be true, or on some condition which suggests the necessary construction. If this attempt is successful, the steps of the argument may in general be re-arranged in reverse order, and the construction and proof presented in a synthetic form.

This unravelling of a proposition in order to trace it back to some earlier principle on which it depends, is called **geometrical analysis** : it is the natural way of attacking many theorems, and it is especially useful in solving *problems.*

Although the above directions do not amount to a *method*, they often furnish a mode of *searching for a suggestion.* Geometrical Analysis however can only be used with success when a thorough grasp of the chief propositions of Euclid has been gained.

The practical application of the foregoing hints is illustrated by the following examples.

1. *Construct an isosceles triangle having given the base, and the sum of one of the equal sides and the perpendicular drawn from the vertex to the base.*



Let AB be the given base, and K the sum of one side and the perpendicular drawn from the vertex to the base.

ANALYSIS. *Suppose* ABC *to be the required triangle.*

From C draw CX perpendicular to AB:
then AB is bisected at X. I. 26.
Now if we produce XC to H, making XH equal to K,
it follows that CH = CA;
and if AH is joined,
we notice that the angle CAH = the angle CHA. I. 5.

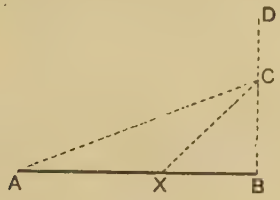
Now the straight lines XH and AH can be drawn *before the position of* C *is known*;

Hence we have the following construction, which we arrange synthetically.

SYNTHESIS. Bisect AB at X:
from X draw XH perpendicular to AB, making XH equal to K.
Join AH.
At the point A in HA, *make the angle* HAC *equal to the angle* AHX.
Join CB.
Then ACB *shall be the triangle required.*

First the triangle is isosceles, for AC = BC. I. 4.
Again, since the angle HAC = the angle AHC, *Constr.*
∴ HC = AC. I. 6.
To each add CX;
then the sum of AC, CX = the sum of HC, CX
= HX.
That is, the sum of AC, CX = K. Q.E.F.

2. *To divide a given straight line so that the square on one part may be double of the square on the other.*



Let AB be the given straight line.

ANALYSIS. *Suppose* AB *to be divided as required at* X: *that is, suppose the square on* AX *to be double of the square on* XB.

Now we remember that in an isosceles right-angled triangle, the square on the hypotenuse is double of the square on either of the equal sides.

This suggests to us to draw BC perpendicular to AB, to make BC equal to BX, and to join XC.

Then the square on XC is double of the square on XB; I. 47.
$\therefore$ XC = AX.

Hence when we join AC, we notice that
the angle XAC = the angle XCA. I. 5.

Thus the exterior angle CXB is double of the angle XAC. I. 32.

But the angle CXB is half of a right angle: I. 32.
$\therefore$ *the angle* XAC *is one-fourth of a right angle.*

This supplies the clue to the following construction:—

SYNTHESIS. From B draw BD perpendicular to AB;
and from A draw AC, *making* BAC *one-fourth of a right angle.*
From C, the intersection of AC and BD, draw CX, making the angle ACX equal to the angle BAC. I. 23.

Then AB *shall be divided as required at* X.

For since the angle XCA = the angle XAC,
$\therefore$ XA = XC. I. 6.

And because the angle BXC = the sum of the angles BAC, ACX, I. 32.
$\therefore$ the angle BXC is half a right angle.

And the angle at B is a right angle;
$\therefore$ the angle BCX is half a right angle; I. 32.
$\therefore$ the angle BXC = the angle BCX;
$\therefore$ BX = BC.

Hence the square on XC is double of the square on XB: I. 47.
that is, the square on AX is double of the square on XB. Q.E.F.

I. ON THE IDENTICAL EQUALITY OF TRIANGLES.

See Propositions 4, 8, 26.

1. If in a triangle the perpendicular from the vertex on the base bisects the base, then the triangle is isosceles.

2. If the bisector of the vertical angle of a triangle is also perpendicular to the base, the triangle is isosceles.

3. If the bisector of the vertical angle of a triangle also bisects the base, the triangle is isosceles.

[Produce the bisector, and complete the construction after the manner of I. 16.]

4. If in a triangle a pair of straight lines drawn from the extremities of the base, making equal angles with the remaining sides, are equal, the triangle is isosceles.

5. If in a triangle the perpendiculars drawn from the extremities of the base to the opposite sides are equal, the triangle is isosceles.

6. Two triangles ABC, ABD on the same base AB, and on opposite sides of it, are such that AC is equal to AD, and BC is equal to BD: shew that the line joining the points C and D is perpendicular to AB.

7. If from the extremities of the base of an isosceles triangle perpendiculars are drawn to the opposite sides, shew that the straight line joining the vertex to the intersection of these perpendiculars bisects the vertical angle.

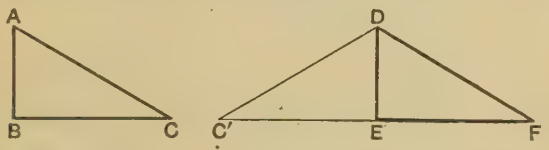
8. ABC is a triangle in which the vertical angle BAC is bisected by the straight line AX: from B draw BD perpendicular to AX, and produce it to meet AC, or AC produced, in E; then shew that BD is equal to DE.

9. In a quadrilateral ABCD, AB is equal to AD, and BC is equal to DC: shew that the diagonal AC bisects each of the angles which it joins.

10. In a quadrilateral ABCD the opposite sides AD, BC are equal, and also the diagonals AC, BD are equal: if AC and BD intersect at K, shew that each of the triangles AKB, DKC is isosceles.

11. If one angle of a triangle be equal to the sum of the other two, the greatest side is double of the distance of its middle point from the opposite angle.

12. *Two right-angled triangles which have their hypotenuses equal, and one side of one equal to one side of the other, are equal in all respects.*



Let ABC, DEF be two $\triangle^s$ right-angled at B and E, having AC equal to DF, and AB equal to DE.

Then shall the $\triangle$ ABC *be equal to the* $\triangle$ DEF *in all respects.*

For apply the $\triangle$ ABC to the $\triangle$ DEF, so that AB may coincide with the equal line DE, and C may fall on the side of DE remote from F. Let C′ be the point on which C falls.

Then DEC′ represents the $\triangle$ ABC in its new position.
Now each of the $\angle^s$ DEF, DEC′ is a rt. $\angle$; *Hyp.*
$\therefore$ EF and EC′ are in one st. line. I. 14.

Then in the $\triangle$ C′DF, because DF = DC′ (*i.e.* AC), *Hyp.*
$\therefore$ the $\angle$ DFC′ = the $\angle$ DC′F. I. 5.

Hence in the two $\triangle^s$ DEF, DEC′,
Because { the $\angle$ DEF = the $\angle$ DEC′, being rt. $\angle^s$;
and the $\angle$ DFE = the $\angle$ DC′E; *Proved.*
also the side DE is common to both;
$\therefore$ the $\triangle^s$ DEF, DEC′ are equal in all respects; I. 26.
that is, the $\triangle^s$ DEF, ABC are equal in all respects. Q.E.D.

Alternative Proof. Since the $\angle$ ABC is a rt. angle;
$\therefore$ the sq. on AC = the sqq. on AB, BC. I. 47.

Similarly, the sq. on DF = the sqq. on DE, EF; I. 47.

But the sq. on AC = the sq. on DF, since AC = DF;
$\therefore$ the sqq. on AB, BC = the sqq. on DE, EF.

And of these, the sq. on AB = the sq. on DE, since AB = DE;
$\therefore$ the sq. on BC = the sq. on EF; *Ax. 3.*
$\therefore$ BC = EF.

Hence the three sides of the $\triangle$ ABC are respectively equal to the three sides of the $\triangle$ DEF;
$\therefore$ the $\triangle$ ABC = the $\triangle$ DEF in all respects. I. 8.

13. *If two triangles have two sides of the one equal to two sides of the other, each to each, and have likewise the angles opposite to one pair of equal sides equal, then the angles opposite to the other pair of equal sides shall be either equal or supplementary, and in the former case the triangles shall be equal in all respects.*

Fig. 1.

Fig. 2.

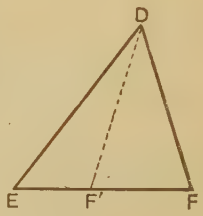


Fig. 3.

Let ABC, DEF be two triangles, in which
the side AB = the side DE,
the side AC = the side DF,
and the ∠ABC = the ∠DEF.

Then shall the ∠ˢ ACB, DFE *be either equal (as in Figs.* 1 *and* 2) *or supplementary (as in Figs.* 1 *and* 3) *; and in the former case the triangles shall be equal in all respects.*

If the ∠BAC = the ∠EDF. [*Figs.* 1 *and* 2.]
then the ∠ACB = the ∠DFE, and the triangles are equal in all respects. I. 4.

But if the ∠BAC be not equal to the ∠EDF, [*Figs.* 1 *and* 3.]
let the ∠EDF be greater than the ∠BAC.

At D in ED make the ∠ EDF′ equal to the ∠ BAC.
Then the △ˢ BAC, EDF′ are equal in all respects. I. 26.
∴ AC = DF′;
but AC = DF; *Hyp.*
∴ DF = DF′,
∴ the angle DFF′ = the ∠ DF′F. I. 5.
But the ∠ˢ DF′F, DF′E are supplementary, I. 13.
∴ the ∠ˢ DFF′, DF′E are supplementary:
that is, the ∠ˢ DFE, ACB are supplementary.

Q.E.D.

Corollaries. Three cases of this theorem deserve special attention.

It has been proved that if the angles ACB, DFE are not *supplementary* they are *equal*:

Hence, in addition to the hypothesis of this theorem,

(i) If the angles ACB, DFE opposite to the two equal sides AB, DE are both acute or both obtuse they cannot be supplementary, and are therefore equal; or if one of them is a right angle, the other must also be a right angle (whether considered as supplementary or equal to it):

in either case the triangles are equal in all respects.

(ii) If the two given angles are right angles or obtuse angles, it follows that the angles ACB, DFE must be both acute, and therefore equal, by (i):

so that the triangles are equal in all respects.

(iii) If in each triangle the side opposite the given angle is not less than the other given side; that is, if AC and DF are not less than AB and DE respectively) then the angles ACB, DFE cannot be greater than the angles ABC, DEF respectively;

therefore the angles ACB, DFE are both acute;

hence, as above, they are equal;

and the triangles ABC, DEF are equal in all respects.

II. ON INEQUALITIES.

See Propositions 16, 17, 18, 19, 20, 21, 24, 25.

1. In a triangle ABC, if AC is not greater than AB, shew that any straight line drawn through the vertex A, and terminated by the base BC, is less than AB.

2. ABC *is a triangle, and the vertical angle* BAC *is bisected by a straight line which meets the base* BC *in* X; *shew that* BA *is greater than* BX, *and* CA *greater than* CX. *Hence obtain a proof of* I. 20.

3. *The perpendicular is the shortest straight line that can be drawn from a given point to a given straight line: and of others, that which is nearer to the perpendicular is less than the more remote; and two, and only two equal straight lines can be drawn from the given point to the given straight line, one on each side of the perpendicular.*

4. The sum of the distances of any point from the three angular points of a triangle is greater than half its perimeter.

5. The sum of the distances of any point within a triangle from its angular points is less than the perimeter of the triangle.

6. The perimeter of a quadrilateral is greater than the sum of its diagonals.

7. The sum of the diagonals of a quadrilateral is less than the sum of the four straight lines drawn from the angular points to any given point. Prove this, and point out the exceptional case.

8. *In a triangle any two sides are together greater than twice the median which bisects the remaining side.* [See Def. p. 79.]

[Produce the median, and complete the construction after the manner of I. 16.]

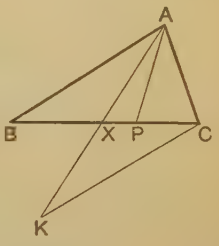
9. *In any triangle the sum of the medians is less than the perimeter.*

10. In a triangle an angle is acute, obtuse, or a right angle, according as the median drawn from it is greater than, less than, or equal to half the opposite side. [See Ex. 4, p. 65.]

11. The diagonals of a rhombus are unequal.

12. *If the vertical angle of a triangle is contained by unequal sides, and if from the vertex the median and the bisector of the angle are drawn, then the median lies within the angle contained by the bisector and the longer side.*

Let ABC be a $\triangle$, in which AB is greater than AC; let AX be the median drawn from A, and AP the bisector of the vertical $\angle$ BAC.



Then shall AX *lie between* AP *and* AB.

Produce AX to K, making XK equal to AX. Join KC.

Then the $\triangle^s$ BXA, CXK may be shewn to be equal in all respects; I. 4.

hence BA = CK, and the $\angle$ BAX = the $\angle$ CKX.

But since BA is greater than AC, *Hyp.*

$\therefore$ CK is greater than AC;

$\therefore$ the $\angle$ CAK is greater than the $\angle$ CKA; I. 18.

that is, the $\angle$ CAX is greater than the $\angle$ BAX:

$\therefore$ the $\angle$ CAX must be more than half the vert. $\angle$ BAC;

hence AX lies within the angle BAP. Q.E.D.

13. *If the vertical angle of a triangle is contained by two unequal sides, and if from the vertex there are drawn the bisector of the vertical angle, the median, and the perpendicular to the base, the first of these lines is intermediate in position and magnitude to the other two.*

III. ON PARALLELS.

See Propositions 27—31.

1. If a straight line meets two parallel straight lines, and the two interior angles on the same side are bisected; shew that the bisectors meet at right angles. [I. 29, I. 32.]

2. The straight lines drawn from any point in the bisector of an angle parallel to the arms of the angle, and terminated by them, are equal; and the resulting figure is a rhombus.

3. AB and CD are two straight lines intersecting at D, and the adjacent angles so formed are bisected: if through any point X in DC a straight line YXZ be drawn parallel to AB and meeting the bisectors in Y and Z, shew that XY is equal to XZ.

4. If two straight lines are parallel to two other straight lines, each to each; and if the acute angles contained by each pair are bisected; shew that the bisecting lines are parallel.

5. The middle point of any straight line which meets two parallel straight lines, and is terminated by them, is equidistant from the parallels.

6. A straight line drawn between two parallels and terminated by them, is bisected; shew that any other straight line passing through the middle point and terminated by the parallels, is also bisected at that point.

7. If through a point equidistant from two parallel straight lines, two straight lines are drawn cutting the parallels, the portions of the latter thus intercepted are equal.

PROBLEMS.

8. AB and CD are two given straight lines, and X is a given point in AB: find a point Y in AB such that YX may be equal to the perpendicular distance of Y from CD.

9. ABC is an isosceles triangle: required to draw a straight line DE parallel to the base BC, and meeting the equal sides in D and E, so that BD, DE, EC may be all equal.

10. ABC is any triangle: required to draw a straight line DE parallel to the base BC, and meeting the other sides in D and E, so that DE may be equal to the sum of BD and CE.

11. ABC is any triangle: required to draw a straight line parallel to the base BC, and meeting the other sides in D and E, so that DE may be equal to the difference of BD and CE.

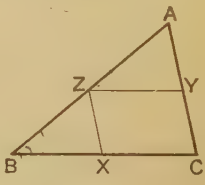
IV. ON PARALLELOGRAMS.

See Propositions 33, 34, and the deductions from these Props. given on page 70.

1. *The straight line drawn through the middle point of a side of a triangle parallel to the base, bisects the remaining side.*

Let ABC be a △, and Z the middle point of the side AB. Through Z, ZY is drawn par[l] to BC.

Then shall Y *be the middle point of* AC.



Through Z draw ZX par[l] to AC. I. 31.

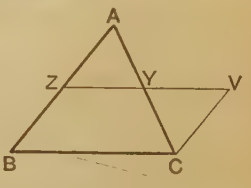
Then in the △[s] AZY, ZBX,
because ZY and BC are par[l],
∴ the ∠ AZY = the ∠ ZBX ; I. 29.
and because ZX and AC are par[l],
∴ the ∠ ZAY = the ∠ BZX ; I. 29.
also AZ = ZB : *Hyp.*
∴ AY = ZX. I. 26.

But ZXCY is a par[m] by construction ;
∴ ZX = YC. I. 34.
Hence AY = YC ;
that is, AC is bisected at Y. Q.E.D.

2. *The straight line which joins the middle points of two sides of a triangle, is parallel to the third side.*

Let ABC be a △, and Z, Y the middle points of the sides AB, AC.

Then shall ZY *be par[l] to* BC.



Produce ZY to V, making YV equal to ZY.

Join CV.
Then in the △[s] AYZ, CYV,
Because { AY = CY, *Hyp.*
and YZ = YV, *Constr.*
and the ∠ AYZ = the vert. opp. ∠ CYV ; I. 15.
∴ AZ = CV, I. 4.
and the ∠ ZAY = the ∠ VCY ;
hence CV is par[l] to AZ. I. 27.
But CV is equal to AZ, that is, to BZ : *Hyp.*
∴ CV is equal and par[l] to BZ :
∴ ZV is equal and par[l] to BC : I. 33.
that is, ZY is par[l] to BC. Q.E.D.

[A second proof of this proposition may be derived from I. 38, 39.]

3. *The straight line which joins the middle points of two sides of a triangle is equal to half the third side.*

4. *Shew that the three straight lines which join the middle points of the sides of a triangle, divide it into four triangles which are identically equal.*

5. *Any straight line drawn from the vertex of a triangle to the base is bisected by the straight line which joins the middle points of the other sides of the triangle.*

6. Given the three middle points of the sides of a triangle, construct the triangle.

7. AB, AC are two given straight lines, and P is a given point between them; required to draw through P a straight line terminated by AB, AC, and bisected by P.

8. ABCD is a parallelogram, and X, Y are the middle points of the opposite sides AD, BC: shew that BX and DY trisect the diagonal AC.

9. *If the middle points of adjacent sides of any quadrilateral are joined, the figure thus formed is a parallelogram.*

10. Shew that the straight lines which join the middle points of opposite sides of a quadrilateral, bisect one another.

11. The straight line which joins the middle points of the oblique sides of a trapezium, is parallel to the two parallel sides, and passes through the middle points of the diagonals.

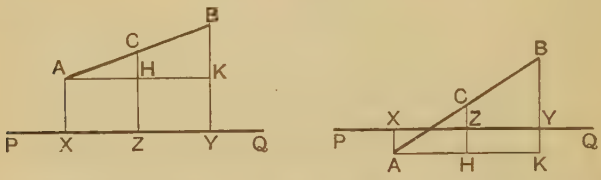
12. *The straight line which joins the middle points of the oblique sides of a trapezium is equal to half the* sum *of the parallel sides; and the portion intercepted between the diagonals is equal to half the* difference *of the parallel sides.*

DEFINITION.

If from the extremities of one straight line perpendiculars are drawn to another, the portion of the latter intercepted between the perpendiculars is said to be the **Orthogonal Projection** of the first line upon the second.

Thus in the adjoining figures, if from the extremities of the straight line AB the perpendiculars AX, BY are drawn to PQ, then XY is the *orthogonal projection* of AB on PQ.

13. *A given straight line* AB *is bisected at* C ; *shew that the projections of* AC, CB *on any other straight line are equal.*



Let XZ, ZY be the projections of AC, CB on any straight line PQ.
Then XZ *and* ZY *shall be equal.*

Through A draw a straight line parallel to PQ, meeting CZ, BY or these lines produced in H, K. I. 31.

Now AX, CZ, BY are parallel, for they are perp. to PQ ; I. 28.
∴ the figures XH, HY are par^{ms} ;
∴ AH = XZ, and HK = ZY. I. 34.

But through C, the middle point of AB, a side of the △ ABK, CH has been drawn parallel to the side BK ;
∴ CH bisects AK : Ex. 1, p. 104.
that is, AH = HK ;
∴ XZ = ZY. Q.E.D.

14. *If three parallel straight lines make equal intercepts on a fourth straight line which meets them, they will also make equal intercepts on any other straight line which meets them.*

15. *Equal and parallel straight lines have equal projections on any other straight line.*

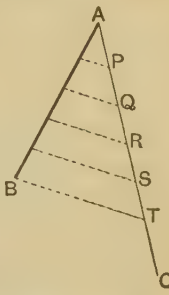
16. AB is a given straight line bisected at O ; and AX, BY are perpendiculars drawn from A and B on any other straight line : shew that OX is equal to OY.

17. AB *is a given straight line bisected at* O : *and* AX, BY *and* OZ *are perpendiculars drawn to any straight line* PQ, *which does not pass between* A *and* B : *shew that* OZ *is equal to half the sum of* AX, BY.

[OZ is said to be the **Arithmetic Mean** between AX and BY.]

18. AB is a given straight line bisected at O ; and through A, B and O parallel straight lines are drawn to meet a given straight line PQ in X, Y, Z : shew that OZ is equal to half the *sum*, or half the *difference* of AX and BY, according as A and B lie on the *same* side or on *opposite* sides of PQ.

19. *To divide a given finite straight line into any number of equal parts.*



[For example: *required to divide the straight line* AB *into five equal parts.*

From A draw AC, a straight line of unlimited length, making any angle with AB.

In AC take *any* point P; and by marking off successive parts PQ, QR, RS, ST each equal to AP, make AT to contain AP *five* times.

Join BT; and through P, Q, R, S draw parallels to BT.

It may be shewn by Ex. 14, p. 106, that these parallels divide AB into five equal parts.]

20. *If through an angle of a parallelogram any straight line is drawn, the perpendicular drawn to it from the opposite angle is equal to the sum or difference of the perpendiculars drawn to it from the two remaining angles, according as the given straight line falls without the parallelogram, or intersects it.*

[Through the opposite angle draw a straight line parallel to the given straight line, so as to meet the perpendicular from one of the remaining angles, produced if necessary; then apply I. 34, I. 26. Or proceed as in the following example.]

21. From the angular points of a parallelogram perpendiculars are drawn to any straight line which is without the parallelogram: shew that the sum of the perpendiculars drawn from one pair of opposite angles is equal to the sum of those drawn from the other pair.

[Draw the diagonals, and from their point of intersection let fall a perpendicular upon the given straight line. See Ex. 17, p. 106.]

22. The sum of the perpendiculars drawn from any point in the base of an isosceles triangle to the equal sides is equal to the perpendicular drawn from either extremity of the base to the opposite side.

[It follows that the sum of the distances of *any* point in the base of an isosceles triangle from the equal sides is **constant**, that is, the same whatever point in the base is taken.]

23. In the base produced of an isosceles triangle any point is taken: shew that the difference of its perpendicular distances from the equal sides is *constant*.

24. The sum of the perpendiculars drawn from any point within an equilateral triangle to the three sides is equal to the perpendicular drawn from any one of the angular points to the opposite side, and is therefore constant.

PROBLEMS.

25. Draw a straight line through a given point, so that the part of it intercepted between two given parallel straight lines may be of given length. When does this problem admit of two solutions, when of only one, and when is it impossible?

26. Draw a straight line parallel to a given straight line, so that the part intercepted between two other given straight lines may be of given length.

27. Draw a straight line equally inclined to two given straight lines that meet, so that the part intercepted between them may be of given length.

28. AB, AC are two given straight lines, and P is a given point *without* the angle contained by them. It is required to draw through P a straight line to meet the given lines, so that the part intercepted between them may be equal to the part between P and the nearer line.

V. MISCELLANEOUS THEOREMS AND EXAMPLES.

Chiefly on I. 32.

1. A *is the vertex of an isosceles triangle* ABC, *and* BA *is produced to* D, *so that* AD *is equal to* BA; *if* DC *is drawn, shew that* BCD *is a right angle.*

2. *The straight line joining the middle point of the hypotenuse of a right-angled triangle to the right angle is equal to half the hypotenuse.*

3. From the extremities of the base of a triangle perpendiculars are drawn to the opposite sides (produced if necessary); shew that the straight lines which join the middle point of the base to the feet of the perpendiculars are equal.

4. *In a triangle* ABC, AD *is drawn perpendicular to* BC; *and* X, Y, Z *are the middle points of the sides* BC, CA, AB *respectively: shew that each of the angles* ZXY, ZDY *is equal to the angle* BAC.

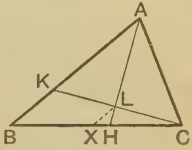
5. *In a right-angled triangle, if a perpendicular is drawn from the right angle to the hypotenuse, the two triangles thus formed are equiangular to one another.*

6. *In a right-angled triangle two straight lines are drawn from the right angle, one bisecting the hypotenuse, the other perpendicular to it: shew that they contain an angle equal to the difference of the two acute angles of the triangle.* [See above, Ex. 2 and Ex. 5.]

7. *In a triangle if a perpendicular is drawn from one extremity of the base to the bisector of the vertical angle,* (i) *it will make with either of the sides containing the vertical angle an angle equal to half the sum of the angles at the base;* (ii) *it will make with the base an angle equal to half the difference of the angles at the base.*

Let ABC be the given △, and AH the bisector of the vertical ∠ BAC.

Let CLK meet AH at right angles.



(i) *Then shall each of the ∠s* AKC, ACK *be equal to half the sum of the ∠s* ABC, ACB.

In the △s AKL, ACL,

Because { the ∠ KAL = the ∠ CAL, *Hyp.*
also the ∠ ALK = the ∠ ALC, being rt. ∠s ;
and AL is common to both △s ;

∴ the ∠ AKL = the ∠ ACL. I. 26.

Again, the ∠ AKC = the sum of the ∠s KBC, KCB ; I. 32.

∴ the ∠ ACK = the sum of the ∠s KBC, KCB.

To each add the ∠ ACK :

then twice the ∠ ACK = the sum of the ∠s ABC, ACB ;

∴ the ∠ ACK = half the sum of the ∠s ABC, ACB.

(ii) *The ∠* KCB *shall be equal to half the difference of the ∠s* ACB, ABC.

As before, the ∠ ACK = the sum of the ∠s KBC, KCB.

To each of these add the ∠ KCB :

then the ∠ ACB = the ∠ KBC together with twice the ∠ KCB.

∴ twice the ∠ KCB = the difference of the ∠s ACB, KBC ;

that is, the ∠ KCB = half the difference of the ∠s ACB, ABC.

COROLLARY. *If* X *is the middle point of the base, and* XL *is joined, it may be shewn by Ex.* 3, *p.* 105, *that* XL *is half* BK ; *that is, that* XL *is half the difference of the sides* AB, AC.

8. *In any triangle the angle contained by the bisector of the vertical angle and the perpendicular from the vertex to the base is equal to half the difference of the angles at the base.* [See Ex. 3, p. 65.]

9. In a triangle ABC the side AC is produced to D, and the angles BAC, BCD are bisected by straight lines which meet at F ; shew that they contain an angle equal to half the angle at B.

10. If in a right-angled triangle one of the acute angles is double of the other, shew that the hypotenuse is double of the shorter side.

11. If in a diagonal of a parallelogram any two points equidistant from its extremities are joined to the opposite angles, the figure thus formed will be also a parallelogram.

12. ABC is a given equilateral triangle, and in the sides BC, CA, AB the points X, Y, Z are taken respectively, so that BX, CY and AZ are all equal. AX, BY, CZ are now drawn, intersecting in P, Q, R: shew that the triangle PQR is equilateral.

13. If in the sides AB, BC, CD, DA of a parallelogram ABCD four points P, Q, R, S are taken in order, one in each side, so that AP, BQ, CR, DS are all equal; shew that the figure PQRS is a parallelogram.

14. In the figure of I. 1, if the circles intersect at F, and if CA and CB are produced to meet the circles in P and Q respectively; shew that the points P, F, Q are in the same straight line; and shew also that the triangle CPQ is equilateral.

[Problems marked (*) admit in general of more than one solution.]

15. Through two given points draw two straight lines forming with a straight line given in position, an equilateral triangle.

*16. From a given point it is required to draw to two parallel straight lines two equal straight lines at right angles to one another.

*17. Three given straight lines meet at a point; draw another straight line so that the two portions of it intercepted between the given lines may be equal to one another.

18. From a given point draw three straight lines of given lengths, so that their extremities may be in the same straight line, and intercept equal distances on that line. [See Fig. to I. 16.]

19. Use the properties of the equilateral triangle to trisect a given finite straight line.

20. In a given triangle inscribe a rhombus, having one of its angles coinciding with an angle of the triangle.

VI. ON THE CONCURRENCE OF STRAIGHT LINES IN A TRIANGLE.

DEFINITIONS. (i) Three or more straight lines are said to be **concurrent** when they meet in one point.

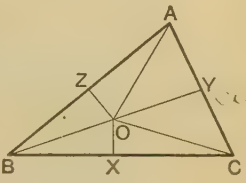
(ii) Three or more points are said to be **collinear** when they lie upon one straight line.

Obs. We here give some propositions relating to the concurrence of certain groups of straight lines drawn in a triangle: the importance of these theorems will be more fully appreciated when the student is familiar with Books III. and IV.

1. *The perpendiculars drawn to the sides of a triangle from their middle points are concurrent.*

Let ABC be a △, and X, Y, Z the middle points of its sides.

Then shall the perp^{s} *drawn to the sides from* X, Y, Z *be concurrent.*



From Z and Y draw perp^{s} to AB, AC; these perp^{s}, since they cannot be parallel, will meet at some point O. *Ax.* 12.

Join OX.

It is required to prove that OX *is perp. to* BC.

Join OA, OB, OC.

In the △^{s} OYA, OYC,

Because { YA = YC, *Hyp.*
and OY is common to both;
also the ∠ OYA = the ∠ OYC, being rt. ∠^{s};
∴ OA = OC. I. 4.

Similarly, from the △^{s} OZA, OZB, it may be proved that OA = OB.

Hence OA, OB, OC are all equal.

Again, in the △^{s} OXB, OXC,

Because { BX = CX, *Hyp.*
and XO is common to both;
also OB = OC: *Proved.*
∴ the ∠ OXB = the ∠ OXC; I. 8.

but these are adjacent ∠^{s};
∴ they are rt. ∠^{s}; *Def.* 10.

that is, OX is perp. to BC.

Hence the three perp^{s} OX, OY, OZ meet at the point O.

Q.E.D.

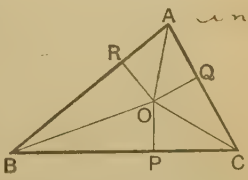
2. *The bisectors of the angles of a triangle are concurrent.*

Let ABC be a △. Bisect the ∠^{s} ABC, BCA, by straight lines which must meet at some point O. *Ax.* 12.

Join AO.

It is required to prove that AO *bisects the* ∠ BAC.

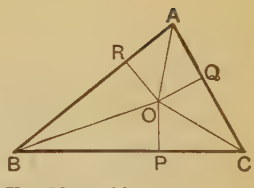
From O draw OP, OQ, OR perp. to the sides of the △.



Then in the △^{s} OBP, OBR,

Because { the ∠ OBP = the ∠ OBR, *Constr.*
and the ∠ OPB = the ∠ ORB, being rt. ∠^{s},
and OB is common;
∴ OP = OR. I. 26.

Similarly from the △s OCP, OCQ,
it may be shewn that OP = OQ,
∴ OP, OQ, OR are all equal.



Again in the △s ORA, OQA,

Because { the ∠s ORA, OQA are rt. ∠s,
and the hypotenuse OA is common,
also OR = OQ; *Proved.*

∴ the ∠ RAO = the ∠ QAO. Ex. 12, p. 99.

That is, AO is the bisector of the ∠ BAC.

Hence the bisectors of the three ∠s meet at the point O.

Q.E.D.

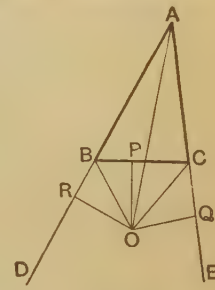
3. *The bisectors of two exterior angles of a triangle and the bisector of the third angle are concurrent.*

Let ABC be a △, of which the sides AB, AC are produced to any points D and E.

Then shall the bisectors of the ∠s DBC, ECB, BAC *be concurrent.*

Bisect the ∠s DBC, ECB by straight lines which must meet at some point O. *Ax.* 12.

Join AO.



It is required to prove that AO *bisects the angle* BAC.

From O draw OP, OQ, OR perp. to the sides of the △.

Then in the △s OBP, OBR,

Because { the ∠ OBP = the ∠ OBR, *Constr.*
also the ∠ OPB = the ∠ ORB, being rt. ∠s,
and OB is common;
∴ OP = OR. I. 26.

Similarly from the △s OCP, OCQ,
it may be shewn that OP = OQ:
∴ OP, OQ, OR are all equal.

Again in the △s ORA, OQA,

Because { the ∠s ORA, OQA are rt. ∠s,
and the hypotenuse OA is common,
also OR = OQ; *Proved.*
∴ the ∠ RAO = the ∠ QAO. Ex. 12, p. 99.

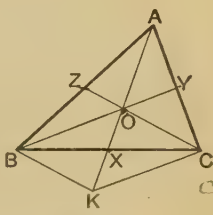
That is, AO is the bisector of the ∠ BAC.

∴ the bisectors of the two exterior ∠s DBC, ECB, and of the interior ∠ BAC meet at the point O.

Q.E.D.

4. *The medians of a triangle are concurrent.*

Let ABC be a △.
Then shall its three medians be concurrent.
Let BY and CZ be two of its medians, and let them intersect at O.
Join AO,
and produce it to meet BC in X.
It is required to shew that AX *is the remaining median of the* △.
Through C draw CK parallel to BY:
produce AX to meet CK at K.
Join BK.



In the △ AKC,
because Y is the middle point of AC, and YO is parallel to CK,
∴ O is the middle point of AK. Ex. 1, p. 104.

Again in the △ ABK,
since Z and O are the middle points of AB, AK,
∴ ZO is parallel to BK, Ex. 2, p. 104.
that is, OC is parallel to BK:
∴ the figure BKCO is a par^{m}.
But the diagonals of a par^{m} bisect one another, Ex. 5, p. 70.
∴ X is the middle point of BC.
That is, AX is a median of the △.

Hence the three medians meet at the point O. Q.E.D.

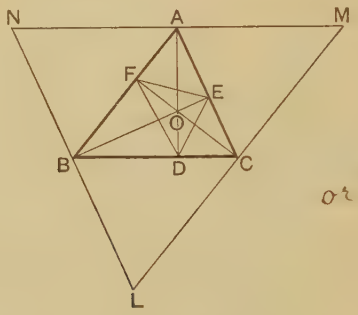
COROLLARY. *The three medians of a triangle cut one another at a point of trisection, the greater segment in each being towards the angular point.*

For in the above figure it has been proved that
AO = OK,
also that OX is half of OK;
∴ OX is half of OA:
that is, OX is one third of AX.
Similarly OY is one third of BY,
and OZ is one third of CZ. Q.E.D.

By means of this Corollary it may be shewn that in any triangle the shorter median bisects the greater side.

[The point of intersection of the three medians of a triangle is called the **centroid**. It is shewn in Mechanics that a thin triangular plate will balance in any position about this point: therefore the centroid of a triangle is also its centre of gravity.]

5. *The perpendiculars drawn from the vertices of a triangle to the opposite sides are concurrent.*



orthocentre.

Let ABC be a △, and AD, BE, CF the three perp^s drawn from the vertices to the opposite sides.

Then shall the perp^s AD, BE, CF *be concurrent.*

Through A, B, and C draw straight lines MN, NL, LM parallel to the opposite sides of the △.

Then the figure BAMC is a par^m. *Def.* 36.
∴ AB = MC. I. 34.
Also the figure BACL is a par^m.
∴ AB = LC,
∴ LC = CM :
that is, C is the middle point of LM.
So also A and B are the middle points of MN and NL.
Hence AD, BE, CF are the perp^s to the sides of the △ LMN from their middle points. Ex. 3, p. 60.

But these perp^s meet in a point : Ex. 1, p. 111.
that is, the perp^s drawn from the vertices of the △ ABC to the opposite sides meet in a point. Q.E.D.

[For another proof see Theorems and Examples on Book III.]

DEFINITIONS.

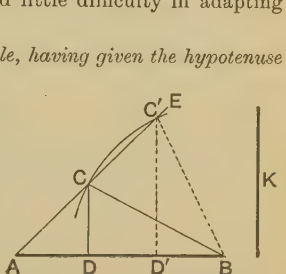
(i) The intersection of the perpendiculars drawn from the vertices of a triangle to the opposite sides is called its **orthocentre.**

(ii) The triangle formed by joining the feet of the perpendiculars is called the **pedal triangle.**

VII. ON THE CONSTRUCTION OF TRIANGLES WITH GIVEN PARTS.

Obs. No general rules can be laid down for the solution of problems in this section; but in a few typical cases we give constructions, which the student will find little difficulty in adapting to other questions of the same class.

1. *Construct a right-angled triangle, having given the hypotenuse and the sum of the remaining sides.*



[It is required to construct a rt.-angled △, having its hypotenuse equal to the given straight line K, and the sum of its remaining sides equal to AB.

From A draw AE making with BA an ∠ equal to half a rt. ∠. From centre B, with radius equal to K, describe a circle cutting AE in the points C, C′.

From C and C′ draw perp^{s} CD, C′D′ to AB; and join CB, C′B. Then either of the △^{s} CDB, C′D′B will satisfy the given conditions.

NOTE. If the given hypotenuse K be greater than the perpendicular drawn from B to AE, there will be *two* solutions. If the line K be equal to this perpendicular, there will be *one* solution; but if less, the problem is *impossible.*]

2. Construct a right-angled triangle, having given the hypotenuse and the difference of the remaining sides.

3. Construct an isosceles right-angled triangle, having given the sum of the hypotenuse and one side.

4. *Construct a triangle, having given the perimeter and the angles at the base.*

[Let AB be the perimeter of the required △, and X and Y the ∠^{s} at the base.

From A draw AP, making the ∠ BAP equal to half the ∠ X.
From B draw BP, making the ∠ ABP equal to half the ∠ Y.
From P draw PQ, making the ∠ APQ equal to the ∠ BAP.
From P draw PR, making the ∠ BPR equal to the ∠ ABP.
Then shall PQR be the required △.]

5. Construct a right-angled triangle, having given the perimeter and one acute angle.

6. Construct an isosceles triangle of given altitude, so that its base may be in a given straight line, and its two equal sides may pass through two fixed points. [See Ex. 7, p. 55.]

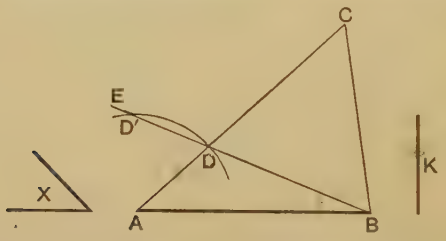
7. Construct an equilateral triangle, having given the length of the perpendicular drawn from one of the vertices to the opposite side.

8. Construct an isosceles triangle, having given the base, and the difference of one of the remaining sides and the perpendicular drawn from the vertex to the base. [See Ex. 1, p. 96.]

9. Construct a triangle, having given the base, one of the angles at the base, and the sum of the remaining sides.

10. Construct a triangle, having given the base, one of the angles at the base, and the difference of the remaining sides. [Two cases arise, according as the given angle is adjacent to the greater side or the less.]

11. *Construct a triangle, having given the base, the difference of the angles at the base, and the difference of the remaining sides.*



[Let AB be the given base, X the difference of the ∠ˢ at the base, and K the difference of the remaining sides.

Draw BE, making the ∠ ABE equal to half the ∠ X.

From centre A, with radius equal to K, describe a circle cutting BE in D and D'. Let D be the point of intersection *nearer to* B.

Join AD and produce it to C.

Draw BC, making the ∠ DBC equal to the ∠ BDC.

Then shall CAB be the △ required. Ex. 7, p. 109.

NOTE. This problem is possible only when the given difference K is greater than the perpendicular drawn from A to BE.]

12. Construct a triangle, having given the base, the difference of the angles at the base, and the sum of the remaining sides.

13. Construct a triangle, having given the perpendicular from the vertex on the base, and the difference between each side and the adjacent segment of the base.

14. Construct a triangle, having given two sides and the median which bisects the remaining side. [See Ex. 18, p. 110.]

15. Construct a triangle, having given one side, and the medians which bisect the two remaining sides.

[See Fig. to Ex. 4, p. 113.
Let BC be the given side. Take two-thirds of each of the given medians; hence construct the triangle BOC. The rest of the construction follows easily.]

16. *Construct a triangle, having given its three medians.*

[See Fig. to Ex. 4, p. 113.
Take two-thirds of each of the given medians, and construct the triangle OKC. The rest of the construction follows easily.]

VIII. ON AREAS.

See Propositions 35—48.

Obs. It must be understood that throughout this section the word *equal* as applied to rectilineal figures will be used as denoting *equality of area* unless otherwise stated.

1. *Shew that a parallelogram is bisected by any straight line which passes through the middle point of one of its diagonals.* [I. 29, 26.]

2. Bisect a parallelogram by a straight line drawn through a given point.

3. Bisect a parallelogram by a straight line drawn perpendicular to one of its sides.

4. Bisect a parallelogram by a straight line drawn parallel to a given straight line.

5. ABCD *is a trapezium in which the side* AB *is parallel to* DC. *Shew that its area is equal to the area of a parallelogram formed by drawing through* X, *the middle point of* BC, *a straight line parallel to* AD, *meeting* DC, *or* DC *produced.* [I. 29, 26.]

6. A trapezium is equal to a parallelogram whose base is half the sum of the parallel sides of the given figure, and whose altitude is equal to the perpendicular distance between them.

7. ABCD is a trapezium in which the side AB is parallel to DC; shew that it is double of the triangle formed by joining the extremities of AD to X, the middle point of BC.

8. Shew that a trapezium is bisected by the straight line which joins the middle points of its parallel sides. [I. 38.]

Obs. In the following group of Exercises the proofs depend chiefly on Propositions 37 and 38, and the two converse theorems.

9. If two straight lines AB, CD intersect at X, and if the straight lines AC and BD, which join their extremities are parallel, shew that the triangle AXD is equal to the triangle BXC.

10. If two straight lines AB, CD intersect at X, so that the triangle AXD is equal to the triangle XCB, then AC and BD are parallel.

11. ABCD is a parallelogram, and X any point in the diagonal AC produced; shew that the triangles XBC, XDC are equal. [See Ex. 13, p. 70.]

12. ABC is a triangle, and R, Q the middle points of the sides AB, AC; shew that if BQ and CR intersect in X, the triangle BXC is equal to the quadrilateral AQXR. [See Ex. 5, p. 79.]

13. If the middle points of the sides of a quadrilateral be joined in order, the *parallelogram* so formed [see Ex. 9, p. 105] is equal to half the given figure.

14. Two triangles of equal area stand on the same base but on opposite sides of it: shew that the straight line joining their vertices is bisected by the base, or by the base produced.

15. The straight line which joins the middle points of the diagonals of a trapezium is parallel to each of the two parallel sides.

16. (i) *A triangle is equal to the sum or difference of two triangles on the same base (or on equal bases), if the altitude of the first is equal to the sum or difference of the altitudes of the others.*

(ii) *A triangle is equal to the sum or difference of two triangles of the same altitude, if the base of the first is equal to the sum or difference of the bases of the others.*

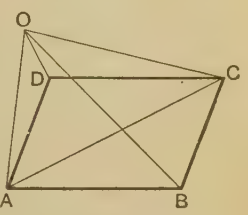
Similar statements hold good of parallelograms.

17. ABCD is a parallelogram, and O is any point outside it; shew that the sum or difference of the triangles OAB, OCD is equal to half the parallelogram. Distinguish between the two cases.

Obs. On the following proposition depends an important theorem in Mechanics: we give a proof of the first case, leaving the second case to be deduced by a similar method.

18. (i) ABCD *is a parallelogram, and* O *is any point* without the angle BAD and its opposite vertical angle; *shew that the triangle* OAC *is equal to the* sum *of the triangles* OAD, OAB.

(ii) *If* O *is* within the angle BAD or its opposite vertical angle, *the triangle* OAC *is equal to the difference of the triangles* OAD, OAB.



Case I. If O is without the ∠ DAB and its opp. vert. ∠, then OA is without the par[m] ABCD: therefore the perp. drawn from C to OA is equal to the sum of the perp[s] drawn from B and D to OA. [See Ex. 20, p. 107.]

Now the △[s] OAC, OAD, OAB are upon the same base OA;

and the altitude of the △ OAC with respect to this base has been shewn to be equal to the sum of the altitudes of the △[s] OAD, OAB.

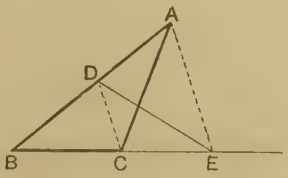
Therefore the △ OAC is equal to the sum of the △[s] OAD, OAB. [See Ex. 16, p. 118.] Q.E.D.

19. ABCD is a parallelogram, and through O, any point within it, straight lines are drawn parallel to the sides of the parallelogram; shew that the difference of the parallelograms DO, BO is double of the triangle AOC. [See preceding theorem (ii).]

20. The area of a quadrilateral is equal to the area of a triangle having two of its sides equal to the diagonals of the given figure, and the included angle equal to either of the angles between the diagonals.

21. ABC *is a triangle, and* D *is any point in* AB; *it is required to draw through* D *a straight line* DE *to meet* BC *produced in* E, *so that the triangle* DBE *may be equal to the triangle* ABC.

Important.



[Join DC. Through A draw AE parallel to DC. I. 31.
Join DE.

The ∠. EBD shall be equal to the △ ABC.] I. 37.

22. On a base of given length describe a triangle equal to a given triangle and having an angle equal to an angle of the given triangle.

23. Construct a triangle equal in area to a given triangle, and having a given altitude.

24. On a base of given length construct a triangle equal to a given triangle, and having its vertex on a given straight line.

25. On a base of given length describe (i) an isosceles triangle; (ii) a right-angled triangle, equal to a given triangle.

26. Construct a triangle equal to the sum or difference of two given triangles. [See Ex. 16, p. 118.]

27. ABC is a given triangle, and X a given point: describe a triangle equal to ABC, having its vertex at X, and its base in the same straight line as BC.

28. ABCD *is a quadrilateral. On the base* AB *construct a triangle equal in area to* ABCD, *and having the angle at* A *common with the quadrilateral.*

[Join BD. Through C draw CX parallel to BD, meeting AD produced in X; join BX.]

29. *Construct a rectilineal figure equal to a given rectilineal figure, and having fewer sides by one than the given figure.*

Hence shew how to construct a triangle equal to a given rectilineal figure.

30. ABCD is a quadrilateral: it is required to construct a triangle equal in area to ABCD, having its vertex at a given point X in DC, and its base in the same straight line as AB.

31. Construct a rhombus equal to a given parallelogram.

32. Construct a parallelogram which shall have the same area and perimeter as a given triangle.

33. *Bisect a triangle by a straight line drawn through one of its angular points.*

34. Trisect a triangle by straight lines drawn through one of its angular points. [See Ex. 19, p. 110, and I. 38.]

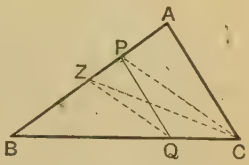
35. Divide a triangle into any number of equal parts by straight lines drawn through one of its angular points.

[See Ex. 19, p. 107, and I. 38.]

36. *Bisect a triangle by a straight line drawn through a given point in one of its sides.*

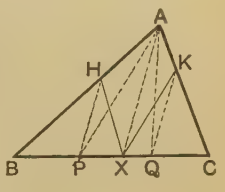
[Let ABC be the given △, and P the given point in the side AB.

Bisect AB at Z; and join CZ, CP.
Through Z draw ZQ parallel to CP.
Join PQ.
Then shall PQ bisect the △.
See Ex. 21, p. 119.]



37. *Trisect a triangle by straight lines drawn from a given point in one of its sides.*

[Let ABC be the given △, and X the given point in the side BC.
Trisect BC at the points P, Q. Ex. 19, p. 107.
Join AX, and through P and Q draw PH and QK parallel to AX.
Join XH, XK.
These straight lines shall trisect the △; as may be shewn by joining AP, AQ.
See Ex. 21, p. 119.]



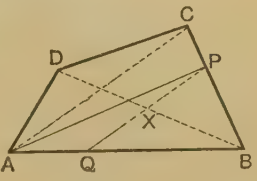
38. Cut off from a given triangle a fourth, fifth, sixth, or any part required by a straight line drawn from a given point in one of its sides. [See Ex. 19, p. 107, and Ex. 21, p. 119.]

39. *Bisect a quadrilateral by a straight line drawn through an angular point.*

[Two constructions may be given for this problem: the first will be suggested by Exercises 28 and 33, p. 120.

The second method proceeds thus.

Let ABCD be the given quadrilateral, and A the given angular point.
Join AC, BD, and bisect BD in X.
Through X draw PXQ parallel to AC, meeting BC in P; join AP.
Then shall AP bisect the quadrilateral.
Join AX, CX, and use I. 37, 38.]



40. Cut off from a given quadrilateral a third, a fourth, a fifth, or any part required, by a straight line drawn through a given angular point. [See Exercises 28 and 35, p. 120.]

Obs. The following Theorems depend on I. 47.

41. In the figure of I. 47, shew that

(i) the sum of the squares on AB and AE is equal to the sum of the squares on AC and AD.

(ii) the square on EK is equal to the square on AB with four times the square on AC.

(iii) the sum of the squares on EK and FD is equal to five times the square on BC.

42. If a straight line is divided into any two parts, the square on the straight line is greater than the sum of the squares on the two parts.

43. If the square on one side of a triangle is less than the squares on the remaining sides, the angle contained by these sides is acute; if greater, obtuse.

44. ABC is a triangle, right-angled at A; the sides AB, AC are intersected by a straight line PQ, and BQ, PC are joined: shew that the sum of the squares on BQ, PC is equal to the sum of the squares on BC, PQ.

45. In a right-angled triangle four times the sum of the squares on the medians which bisect the sides containing the right angle is equal to five times the square on the hypotenuse.

46. Describe a square whose area shall be three times that of a given square.

47. Divide a straight line into two parts such that the sum of their squares shall be equal to a given square.

IX. ON LOCI.

In many geometrical problems we are required to find the position of a point which satisfies given conditions; and all such problems hitherto considered have been found to admit of a *limited number* of solutions. This, however, will not be the case if *only one* condition is given. For example:

†(i) *Required a point which shall be at a given distance from a given point.*

This problem is evidently *indeterminate*, that is to say, it admits of an indefinite number of solutions; for the condition stated is satisfied by *any* point on the circumference of the circle described from the given point as centre, with a radius equal to the given distance. Moreover this condition is satisfied by no other point within or without the circle.

(ii) *Required a point which shall be at a given distance from a given straight line.*

Here again there are an infinite number of such points, and they lie on two parallel straight lines drawn on either side of the given straight line at the given distance from it: further, no point that is not on one or other of these parallels satisfies the given condition.

Hence we see that *one* condition is not sufficient to determine the position of a point absolutely, but it may have the effect of restricting it to some definite line or lines, straight or curved. This leads us to the following definition.

DEFINITION. The **Locus** of a point satisfying an assigned condition consists of the line, lines, or part of a line, to which the point is thereby restricted; provided that the condition is satisfied by every point on such line or lines, and by no other.

A locus is sometimes defined as the path traced out by a point which moves in accordance with an assigned law.

Thus the locus of a point, which is always at a given distance from a given point, is a circle of which the given point is the centre: and the locus of a point, which is always at a given distance from a given straight line, is a pair of parallel straight lines.

We now see that in order to infer that a certain line, or system of lines, is the locus of a point under a given condition, it is necessary to prove

(i) that any point which fulfils the given condition is on the supposed locus;

(ii) that every point on the supposed locus satisfies the given condition.

1. *Find the locus of a point which is always equidistant from two given points.*

Let A, B be the two given points.

(*a*) Let P be any point equidistant from A and B, so that AP = BP.

Bisect AB at X, and join PX.

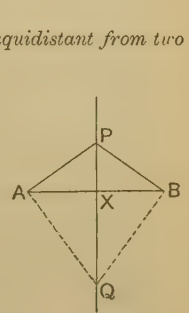
Then in the $\triangle^s$ AXP, BXP,

Because { AX = BX, *Constr.*
and PX is common to both,
also AP = BP, *Hyp.*

$\therefore$ the $\angle$ PXA = the $\angle$ PXB; I. 8.

and they are adjacent $\angle^s$;

$\therefore$ PX is perp. to AB. *Def.* 10.



$\therefore$ any point which is equidistant from A and B is on the straight line which bisects AB at right angles.

(β) Also every point in this line is equidistant from A and B.

For let Q be any point in this line.

Join AQ, BQ.

Then in the $\triangle^s$ AXQ, BXQ,

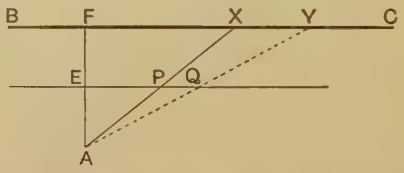
Because { AX = BX,
and XQ is common to both;
also the $\angle$ AXQ = the $\angle$ BXQ, being rt. $\angle^s$;

$\therefore$ AQ = BQ. I. 4.

That is, Q is equidistant from A and B.

Hence we conclude that the locus of the point equidistant from two given points A, B is the straight line which bisects AB at right angles.

2. *To find the locus of the middle point of a straight line drawn from a given point to meet a given straight line of unlimited length.*



Let A be the given point, and BC the given straight line of unlimited length.

(α) Let AX be any straight line drawn through A to meet BC, and let P be its middle point.

Draw AF perp. to BC, and bisect AF at E.

Join EP, and produce it indefinitely.

Since AFX is a $\triangle$, and E, P the middle points of the two sides AF, AX,

$\therefore$ EP is parallel to the remaining side FX. Ex. 2, p. 104.

$\therefore$ P is on the straight line which passes through the *fixed point* E, and is parallel to BC.

(β) Again, every point in EP, or EP produced, fulfils the required condition.

For, in this straight line take any point Q.

Join AQ, and produce it to meet BC in Y.

Then FAY is a $\triangle$, and through E, the middle point of the side AF, EQ is drawn parallel to the side FY;

$\therefore$ Q is the middle point of AY. Ex. 1, p. 104.

Hence the required locus is the straight line drawn parallel to BC, and passing through E, the middle point of the perp. from A to BC.

3. *Find the locus of a point equidistant from two given intersecting straight lines.* [See Ex. 3, p. 55.]

4. Find the locus of a point at a given radial distance from the circumference of a given circle.

5. Find the locus of a point which moves so that the sum of its distances from two given intersecting straight lines of unlimited length is constant.

6. Find the locus of a point when the differences of its distances from two given intersecting straight lines of unlimited length is constant.

7. A straight rod of given length slides between two straight rulers placed at right angles to one another: find the locus of its middle point. [See Ex. 2, p. 108.]

8. On a given base as hypotenuse right-angled triangles are described: find the locus of their vertices. [See Ex. 2, p. 108.]

9. AB is a given straight line, and AX is the perpendicular drawn from A to any straight line passing through B: find the locus of the middle point of AX.

10. Find the locus of the vertex of a triangle, when the base and area are given.

11. Find the locus of the intersection of the diagonals of a parallelogram, of which the base and area are given.

12. Find the locus of the intersection of the medians of triangles described on a given base and of given area.

X. ON THE INTERSECTION OF LOCI.

It appears from various problems which have already been considered, that we are often required to find a point, the position of which is subject to two given conditions. The method of loci is very useful in solving problems of this kind; for corresponding to each condition there will be a locus on which the required point must lie. Hence all points which are common to these two loci, that is, all the points of intersection of the loci, will satisfy *both* the given conditions.

EXAMPLE 1. *To construct a triangle, having given the base, the altitude, and the length of the median which bisects the base.*

Let AB be the given base, and P and Q the lengths of the altitude and median respectively:

then the triangle is known if its *vertex* is known.

(i) Draw a straight line CD parallel to AB, and at a distance from it equal to P:

then the required vertex must lie on CD.

(ii) Again, from the middle point of AB as centre, with radius equal to Q, describe a circle:

then the required vertex must lie on this circle.

Hence any points which are common to CD and the circle, satisfy both the given conditions: that is to say, if CD intersect the circle in E, F each of the points of intersection might be the vertex of the required triangle. This supposes the length of the median Q to be greater than the altitude.

EXAMPLE 2. *To find a point equidistant from three given points* A, B, C, *which are not in the same straight line.*

(i) The locus of points equidistant from A and B is the straight line PQ, which bisects AB at right angles. Ex. 1, p. 123.

(ii) Similarly the locus of points equidistant from B and C is the straight line RS which bisects BC at right angles.

Hence the point common to PQ and RS must satisfy both conditions: that is to say, the point of intersection of PQ and RS will be equidistant from A, B, and C.

Obs. These principles may also be used to prove the theorems relating to concurrency already given on page 111.

EXAMPLE. *To prove that the bisectors of the angles of a triangle are concurrent.*

Let ABC be a triangle.

Bisect the $\angle^s$ ABC, BCA by straight lines BO, CO: these must meet at some point O. *Ax.* 12.

Join OA.

Then shall OA *bisect the* $\angle$ BAC.

Now BO is the locus of points equidistant from BC, BA; Ex. 3, p. 55.

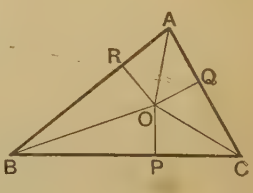
$\therefore$ OP = OR.

Similarly CO is the locus of points equidistant from BC, CA.

$\therefore$ OP = OQ; hence OR = OQ.

$\therefore$ O is on the locus of points equidistant from AB and AC: that is, OA is the bisector of the $\angle$ BAC.

Hence the bisectors of the three $\angle^s$ meet at the point O.



It may happen that the data of the problem are so related to one another that the resulting loci do not intersect. In this case the problem is impossible.

For example, if in Ex. 1, page 126, the length of the given median *is less than* the given altitude, the straight line CD will not be intersected by the circle, and no triangle can fulfil the conditions of the problem. If the length of the median *is equal* to the given altitude, *one* point is common to the two loci; and consequently only one solution of the problem exists: and we have seen that there are two solutions, if the median is greater than the altitude.

In examples of this kind the student should make a point of investigating the relations which must exist among the data, in order that the problem may be possible; and he must observe that if under certain relations *two* solutions are possible, and under other relations no solution exists, there will always be some *intermediate* relation under which *one* and *only one* solution is possible.

EXAMPLES.

1. Find a point in a given straight line which is equidistant from two given points.

2. Find a point which is at given distances from each of two given straight lines. How many solutions are possible?

3. *On a given base construct a triangle, having given one angle at the base and the length of the opposite side. Examine the relations which must exist among the data in order that there may be two solutions, one solution, or that the problem may be impossible.*

4. On the base of a given triangle construct a second triangle equal in area to the first, and having its vertex in a given straight line.

5. Construct an isosceles triangle equal in area to a given triangle, and standing on the same base.

6. Find a point which is at a given distance from a given point, and is equidistant from two given parallel straight lines.

When does this problem admit of two solutions, when of one only, and when is it impossible?

BOOK II.

BOOK II. deals with the areas of rectangles and squares.

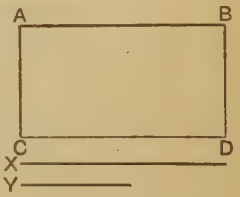
A **Rectangle** has been defined (Book I., Def. 37) as a parallelogram which has one of its angles a right angle.

It should be remembered that if a parallelogram has *one* right angle, *all* its angles are right angles. [I. 46, *Cor.*]

DEFINITIONS.

1. A rectangle is said to be **contained** by any two of its sides which form a right angle: for it is clear that both the form and magnitude of a rectangle are fully determined when the lengths of two such sides are given.

Thus the rectangle ACDB is said *to be contained* by AB, AC; or by CD, DB: and if X and Y are two straight lines equal respectively to AB and AC, then the rectangle contained by X and Y is equal to the rectangle contained by AB, AC.



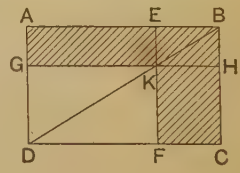
[See Ex. 12, p. 70.]

After Proposition 3, we shall use the abbreviation *rect.* AB, AC to denote *the rectangle contained by* AB *and* AC.

2. In any parallelogram the figure formed by either of the parallelograms about a diagonal together with the two complements is called a **gnomon**.

Thus the shaded portion of the annexed diagram, consisting of the parallelogram EH together with the complements AK, KC is the *gnomon* AHF.

The other gnomon in the diagram is that which is made up of the figures AK, GF and FH, namely the gnomon AFH.

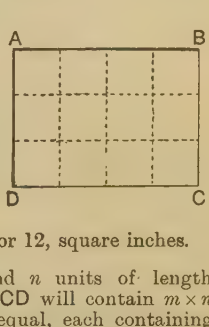


INTRODUCTORY.

Before entering upon Book II. the student is reminded of the following arithmetical rule:

RULE. *To find the area of a rectangle, multiply the number of units in the* **length** *by the number of units in the* **breadth**; *the product will be the number of* **square units** *in the* **area**.

For example, if the two sides AB, AD of the rectangle ABCD are respectively *four* and *three* inches long, and if through the points of division parallels are drawn as in the annexed figure, it is seen that the rectangle is divided into *three rows*, each containing *four* square inches, or into *four columns*, each containing *three* square inches.



Hence the whole rectangle contains 3×4, or 12, square inches.

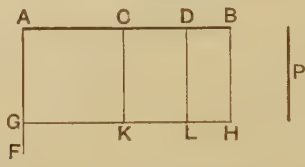
Similarly if AB and AD contain m and n units of length respectively, it follows that the rectangle ABCD will contain $m \times n$ units of area: further, if AB and AD are equal, each containing m units of length, the rectangle becomes a square, and contains m^2 units of area.

From this we conclude that *the rectangle contained by two straight lines* in Geometry corresponds to *the product of two numbers* in Arithmetic or Algebra; and that *the square described on a straight line* corresponds to *the square of a number*. Accordingly it will be found in the course of Book II. that several theorems relating to the areas of rectangles and squares are analogous to well-known algebraical formulæ.

In view of these principles the rectangle contained by two straight lines AB, BC is sometimes expressed in the form of a product, as AB . BC, and the square described on AB as AB^2. This notation, together with the signs + and −, will be employed in the additional matter appended to this book; *but it is not admitted into Euclid's text* because it is desirable in the first instance to emphasize the distinction between *geometrical magnitudes themselves* and the *numerical equivalents* by which they may be expressed arithmetically.

PROPOSITION 1. THEOREM.

If there are two straight lines, one of which is divided into any number of parts, the rectangle contained by the two straight lines is equal to the sum of the rectangles contained by the undivided straight line and the several parts of the divided line.



Let P and AB be two straight lines, and let AB be divided into any number of parts AC, CD, DB.

Then shall the rectangle contained by P, AB *be equal to the sum of the rectangles contained by* P, AC, *by* P, CD, *and by* P, DB.

Construction. From A draw AF perp. to AB; I. 11.
and make AG equal to P. I. 3.
Through G draw GH par^{l} to AB; I. 31.
and through C, D, B draw CK, DL, BH par^{l} to AG.

Proof. Now the fig. AH is made up of the figs. AK, CL, DH, and is therefore equal to their sum;

and of these,

the fig. AH is the rectangle contained by P, AB;
for it is contained by AG, AB; and AG = P:
and the fig. AK is the rectangle contained by P, AC;
for it is contained by AG, AC; and AG = P:
also the fig. CL is the rectangle contained by P, CD;
for it is contained by CK, CD;
and CK = the opp. side AG, and AG = P. I. 34.
Similarly the fig. DH is the rectangle contained by P, DB.

∴ the rectangle contained by P, AB is equal to the sum of the rectangles contained by P, AC, by P, CD, and by P, DB. Q.E.D.

CORRESPONDING ALGEBRAICAL FORMULA.

In accordance with the principles explained on page 129, the result of this proposition may be written thus :

$$\mathsf{P.AB = P.AC + P.CD + P.DB.}$$

Now if the line P contains p units of length, and if AC, CD, DB contain a, b, c units respectively,

$$\text{then } \mathsf{AB} = a+b+c\,;$$

hence the statement

$$\mathsf{P.AB = P.AC + P.CD + P.DB}$$

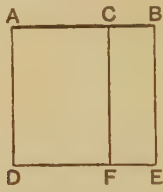
becomes $$p(a+b+c) = pa+pb+pc.$$

[NOTE. It must be understood that the rule given on page 129, for expressing the area of a rectangle as the product of the lengths of two adjacent sides, implies that those sides are **commensurable**, that is, that they can be expressed *exactly* in terms of some common unit.

This however is not always the case. Two straight lines may be so related that it is impossible to divide either of them into equal parts, *of which the other contains an exact number*. Such lines are said to be **incommensurable**. Hence if the adjacent sides of a rectangle are incommensurable, we cannot choose any linear unit in terms of which these sides may be *exactly* expressed ; and thus it will be impossible to subdivide the rectangle into squares of unit area, as illustrated in the figure of page 129. We do not here propose to enter further into the subject of incommensurable quantities : it is sufficient to point out that further knowledge of them will convince the student that the area of a rectangle may be expressed *to any required degree of accuracy* by the product of the lengths of two adjacent sides, whether those lengths are commensurable or not.]

Proposition 2. Theorem.

If a straight line is divided into any two parts, the square on the whole line is equal to the sum of the rectangles contained by the whole line and each of the parts.



Let the straight line AB be divided at C into the two parts AC, CB.

Then shall the square on AB *be equal to the sum of the rectangles contained by* AB, AC, *and by* AB, BC.

Construction. On AB describe the square ADEB. I. 46.
Through C draw CF par[l] to AD. I. 31.

Proof. Now the fig. AE is made up of the figs. AF, CE: and of these,

the fig. AE is the sq. on AB: *Constr.*

and the fig. AF is the rectangle contained by AB, AC; for it is contained by AD, AC; and AD = AB:

also the fig. CE is the rectangle contained by AB, BC; for it is contained by BE, BC; and BE = AB.

∴ the sq. on AB = the sum of the rectangles contained by AB, AC, and by AB, BC. Q.E.D.

Corresponding Algebraical Formula.

The result of this proposition may be written

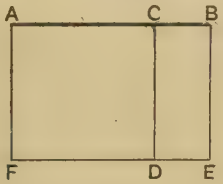
$$AB^2 = AB \cdot AC + AB \cdot BC.$$

Let AC contain a units of length, and let CB contain b units, then $AB = a + b$ units;

and we have $(a+b)^2 = (a+b)a + (a+b)b.$

PROPOSITION 3. THEOREM.

If a straight line is divided into any two parts, the rectangle contained by the whole and one of the parts is equal to the square on that part together with the rectangle contained by the two parts.



Let the straight line AB be divided at C into the two parts AC, CB.

Then shall the rectangle contained by AB, AC *be equal to the square on* AC *together with the rectangle contained by* AC, CB.

Construction. On AC describe the square AFDC. I. 46. Through B draw BE par^{l} to AF, meeting FD produced in E. I. 31.

Proof. Now the fig. AE is made up of the figs. AD, CE; and of these,

the fig. AE is the rectangle contained by AB, AC;
for AF = AC;
and the fig. AD is the sq. on AC; *Constr.*
also the fig. CE is the rectangle contained by AC, CB;
for CD = AC.

∴ the rectangle contained by AB, AC is equal to the sq. on AC together with the rectangle contained by AC, CB.

Q.E.D.

CORRESPONDING ALGEBRAICAL FORMULA.

This result may be written $AB \cdot AC = AC^2 + AC \cdot CB$.

Let AC, CB contain a and b units of length respectively,

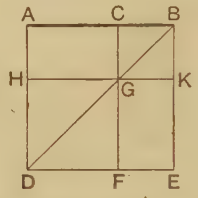
then $AB = a + b$ units;

and we have $(a+b)a = a^2 + ab$.

NOTE. It should be observed that Props. 2 and 3 are *special cases* of Prop. 1.

PROPOSITION 4. THEOREM.

If a straight line is divided into any two parts, the square on the whole line is equal to the sum of the squares on the two parts together with twice the rectangle contained by the two parts.



Let the straight line AB be divided at C into the two parts AC, CB.

Then shall the sq. on AB *be equal to the sum of the sqq. on* AC, CB, *together with twice the rect.* AC, CB.

Construction. On AB describe the square ADEB. I. 46.
Join BD.
Through C draw CF par[l] to BE, meeting BD in G. I. 31.
Through G draw HGK par[l] to AB.

It is first required to shew that the fig. CK is the sq. on CB.

Proof. Because CF and AD are par[l], and BD meets them,
∴ the ext. angle CGB = the int. opp. angle ADB. I. 29.
And since AB = AD, being sides of a square;
∴ the angle ADB = the angle ABD; I. 5.
∴ the angle CGB = the angle CBG.
∴ CB = CG. I. 6.
And the opp. sides of the par[m] CK are equal; I. 34.
∴ the fig. CK is equilateral;
also the angle CBK is a right angle; *Def.* 30.
∴ CK is a square, and it is described on CB. I. 46, *Cor.*

Similarly, the fig. HF is the sq. on HG, that is, the sq. on AC;
for HG = the opp. side AC. I. 34.

Again, the complement AG = the complement GE ; I. 43.
and the fig. AG = the rect. AC, CB ; for CG = CB.
∴ the two figs. AG, GE = twice the rect. AC, CB.

* Now the sq. on AB = the fig. AE
= the figs. HF, CK, AG, GE
= the sqq. on AC, CB together with twice the rect. AC, CB.
∴ the sq. on AB = the sum of the sqq. on AC, CB with twice the rect. AC, CB. Q.E.D.

COROLLARY 1. *Parallelograms about the diagonals of a square are themselves squares.*

COROLLARY 2. *If a straight line is bisected, the square on the whole line is four times the square on half the line.*

* For the purpose of oral work, this step of the proof may conveniently be arranged as follows:

Now the sq. on AB is equal to the fig. AE,
that is, to the figs. HF, CK, AG, GE ;
that is, to the sqq. on AC, CB together with twice the rect. AC, CB.

CORRESPONDING ALGEBRAICAL FORMULA.

The result of this important Proposition may be written thus:

$$AB^2 = AC^2 + CB^2 + 2AC \cdot CB.$$

Let $AC = a$, and $CB = b$;

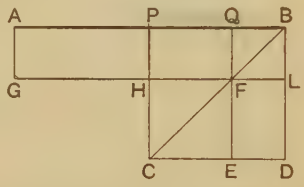
then $AB = a + b$;

hence the statement $AB^2 = AC^2 + CB^2 + 2AC \cdot CB$

becomes $(a+b)^2 = a^2 + b^2 + 2ab.$

PROPOSITION 5. THEOREM.

If a straight line is divided equally and also unequally, the rectangle contained by the unequal parts, together with the square on the line between the points of section, is equal to the square on half the line.



Let the straight line AB be divided equally at P, and unequally at Q.

Then the rect. AQ, QB *together with the sq. on* PQ *shall be equal to the sq. on* PB.

Construction. On PB describe the square PCDB. I. 46.
Join BC.
Through Q draw QE par[l] to BD, cutting BC in F. I. 31.
Through F draw LFHG par[l] to AB.
Through A draw AG par[l] to BD.

Proof.
Now the complement PF = the complement FD: I. 43.
to each add the fig. QL;
then the fig. PL = the fig. QD.

But the fig. PL = the fig. AH, for they are par[ms] on equal bases and between the same par[ls]; I. 36.

∴ the fig. AH = the fig. QD.
To each add the fig. PF;
then the fig. AF = the gnomon PLE.

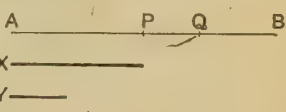
Now the fig. AF is the rect. AQ, QB; for QF = QB;
∴ the rect. AQ, QB = the gnomon PLE.
To each add the sq. on PQ, that is, the fig. HE; II. 4.
then the rect. AQ, QB with the sq. on PQ
= the gnomon PLE with the fig. HE
= the whole fig. PD,
which is the sq. on PB.

That is, the rect. AQ, QB together with the square on PQ is equal to the sq. on PB. Q.E.D.

COROLLARY. From this Proposition it follows that *the difference of the squares on two straight lines is equal to the rectangle contained by their sum and difference.*

For let X and Y be the given st. lines, of which X is the greater.

Draw AP equal to X, and produce it to B, making PB equal to AP, that is to X.



From PB cut off PQ equal to Y.

Then AQ is equal to the sum of X and Y,

and QB is equal to the difference of X and Y.

Now because AB is divided equally at P and unequally at Q,

∴ the rect. AQ, QB with sq. on PQ = the sq. on PB ; II. 5.

that is, the difference of the sqq. on PB, PQ = the rect. AQ, QB.

or, the difference of the sqq. on X and Y = the rectangle contained by the sum and the difference of X and Y.

CORRESPONDING ALGEBRAICAL FORMULA.

This result may be written

$$AQ \,.\, QB + PQ^2 = PB^2.$$

Let $AB = 2a$; and let $PQ = b$;

then AP and PB each $= a$.

Also $AQ = a + b$; and $QB = a - b$.

Hence the statement $AQ \,.\, QB + PQ^2 = PB^2$

becomes $(a+b)(a-b) + b^2 = a^2$,

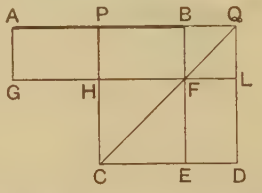
or $(a+b)(a-b) = a^2 - b^2$.

EXERCISE.

In the above figure shew that AP *is half the sum of* AQ *and* QB ; *and that* PQ *is half their difference.*

PROPOSITION 6. THEOREM.

If a straight line is bisected and produced to any point, the rectangle contained by the whole line thus produced and the part of it produced, together with the square on half the line bisected, is equal to the square on the straight line made up of the half and the part produced.



Let the straight line AB be bisected at P, and produced to Q.

Then the rect. AQ, QB *together with the sq. on* PB *shall be equal to the sq. on* PQ.

Construction. On PQ describe the square PCDQ. I. 46.
Join QC.
Through B draw BE par[l] to QD, meeting QC in F. I. 31.
Through F draw LFHG par[l] to AQ.
Through A draw AG par[l] to QD.

Proof. Now the complement PF = the complement FD. I. 43.
But the fig. PF = the fig. AH; for they are par[ms] on equal bases and between the same par[ls]. I. 36.
∴ the fig. AH = the fig. FD.
To each add the fig. PL;
then the fig. AL = the gnomon PLE.
Now the fig. AL is the rect. AQ, QB; for QL = QB;
∴ the rect. AQ, QB = the gnomon PLE.
To each add the sq. on PB, that is, the fig. HE;
then the rect. AQ, QB with the sq. on PB
= the gnomon PLE with the fig. HE
= the whole fig. PD,
which is the square on PQ.

That is, the rect. AQ, QB together with the sq. on PB is equal to the sq. on PQ. Q.E.D.

CORRESPONDING ALGEBRAICAL FORMULA.

This result may be written

$$AQ \cdot QB + PB^2 = PQ^2.$$

Let $AB = 2a$; and let $PQ = b$;

then AP and PB each $= a$.

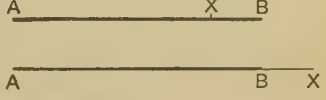
Also $AQ = a + b$; and $QB = b - a$.

Hence the statement $AQ \cdot QB + PB^2 = PQ^2$

becomes $(a+b)(b-a) + a^2 = b^2$,

or $(b+a)(b-a) = b^2 - a^2$.

DEFINITION. If a point X is taken in a straight line AB, or in AB produced, the distances of the point of section from the extremities of AB are said to be the **segments** into which AB is divided at X.



In the former case AB is divided **internally**, in the latter case **externally**.

Thus in *each* of the annexed figures, the segments into which AB is divided at X are the lines AX and XB.

This definition enables us to include Props. 5 and 6 in a single Enunciation.

If a straight line is bisected, and also divided (internally or externally) into two unequal segments, the rectangle contained by the unequal segments is equal to the difference of the squares on half the line, and on the line between the points of section.

EXERCISE.

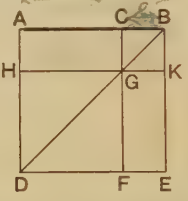
Shew that the Enunciations of Props. 5 and 6 may take the following form:

The rectangle contained by two straight lines is equal to the difference of the squares on half their sum and on half their difference.

[See Ex., p. 137.]

Proposition 7. Theorem.

If a straight line is divided into any two parts, the sum of the squares on the whole line and on one of the parts is equal to twice the rectangle contained by the whole and that part, together with the square on the other part.



Let the straight line AB be divided at C into the two parts AC, CB.

Then shall the sum of the sqq. on AB, BC *be equal to twice the rect.* AB, BC *together with the sq. on* AC.

Construction. On AB describe the square ADEB. I. 46.
Join BD.
Through C draw CF par^{l} to BE, meeting BD in G. I. 31.
Through G draw HGK par^{l} to AB.

Proof. Now the complement AG = the complement GE; I. 43.
to each add the fig. CK:
then the fig. AK = the fig. CE.
But the fig. AK is the rect. AB, BC; for BK = BC;
∴ the two figs AK, CE = twice the rect. AB, BC.
But the two figs. AK, CE make up the gnomon AKF and the fig. CK:
∴ the gnomon AKF with the fig. CK = twice the rect. AB, BC.
To each add the fig. HF, which is the sq. on AC:
then the gnomon AKF with the figs. CK, HF
= twice the rect. AB, BC with the sq. on AC.
But the gnomon AKF with the figs. CK, HF make up the figs. AE, CK, that is to say, the sqq. on AB, BC;
∴ the sqq. on AB, BC = twice the rect. AB, BC with the sq. on AC. Q.E.D.

CORRESPONDING ALGEBRAICAL FORMULA.

The result of this proposition may be written

$$AB^2+BC^2=2AB\,.\,BC+AC^2.$$

Let $AB=a$, and $BC=b$; then $AC=a-b$.

Hence the statement

$$AB^2+BC^2=2AB\,.\,BC+AC^2$$

becomes $$a^2+b^2=2ab+(a-b)^2,$$

or $$(a-b)^2=a^2-2ab+b^2.$$

Comparing this result with that obtained from Prop. 4, we see that

(i) *The square on the* sum *of two straight lines is* greater *than the sum of the squares on those lines by twice the rectangle contained by them.* [Prop. 4.]

(ii) *The square on the* difference *of two straight lines is* less *than the sum of the squares on those lines by twice the rectangle contained by them.* [Prop. 7.]

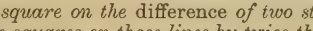
ALTERNATIVE PROOFS OF PROPOSITIONS 4, 5, 6, 7.

The following alternative proofs are recommended for purposes of revision, as affording useful exercise on the enunciations of preceding propositions, and illustrating the way in which many examples on Book II. may be solved. The beginner however should not adopt these proofs until he has thoroughly mastered those given in the text, where the rectangles and squares are actually represented in the diagrams.

PROPOSITION 4.

Let the straight line AB be divided at C into two parts AC, CB.

Then shall the sq. on AB *be equal to the sum of the sqq. on* AC, CB *with twice the rect.* AC, CB.



Now the sq. on AB = the rect. AB, AC with the rect. AB, CB. II. 2.

But the rect. AB, AC = the sq. on AC with the rect. AC, CB; II. 3.

and the rect. AB, CB = the sq. on CB with the rect. AC, CB. II. 3.

Hence the sq. on AB = the sum of the sqq. on AC, CB with twice the rect. AC, CB.

Proposition 5.

Let the straight line AB be divided equally at P, and unequally at Q.

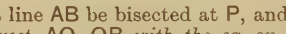
Then shall the rect. AQ, QB *with the sq. on* PQ *be equal to the sq. on* PB.

Now the rect. AQ, QB = the rect. AP, QB with the rect. PQ, QB, II. 1.
= the rect. PB, QB with the rect. PQ, QB.
But the rect. PB, QB = the sq. on QB with the rect. PQ, QB; II. 3.
∴ the rect. AQ, QB = the sq. on QB with *twice* the rect. PQ, QB.
To each of these equals add the sq. on PQ.
Then the rect. AQ, QB with the sq. on PQ
= the sqq. on PQ, QB with twice the rect. PQ, QB
= the sq. on PB. II. 4.

Proposition 6.

Let the straight line AB be bisected at P, and produced to Q.

Then shall the rect. AQ, QB *with the sq. on* PB *be equal to the sq. on* PQ.

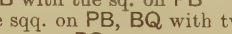


Now the rect. AQ, QB = the rect. AP, BQ with the rect. PQ, BQ II. 1.
= the rect. PB, BQ with the rect. PQ, BQ.
But the rect. PQ, BQ = the sq. on BQ with the rect. PB, BQ. II. 3.
∴ the rect. AQ, QB = the sq. on BQ with *twice* the rect. PB, BQ.
To each of these equals add the sq. on PB.
Then the rect. AQ, QB with the sq. on PB
= the sqq. on PB, BQ with twice the rect. PB, BQ
= the sq. on PQ. II. 4.

Proposition 7.

Let the straight line AB be divided at any point C.

Then shall the sum of the sqq. on AB, BC *be equal to twice the rect.* AB, BC *with the sq. on* AC.

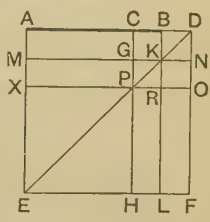


Now the sq. on AB = the sqq. on AC, CB with twice the rect. AC, CB. II. 4.
To each of these equals add the sq. on BC.
Then the sqq. on AB, BC = the sq. on AC with *twice* the sq. on BC and *twice* the rect. AC, CB.
But *twice* the sq. on BC with *twice* the rect. AC, CB
= *twice* the rect. AB, BC. II. 3.
∴ the sqq. on AB, BC = the sq. on AC with twice the rect. AB, BC.

Obs. The following proposition being little used, we merely give the figure and the leading points of Euclid's proof.

Proposition 8. Theorem.

If a straight line is divided into any two parts, four times the rectangle contained by the whole line and one of the parts, together with the square on the other part, is equal to the square on the straight line which is made up of the whole and the first named part.



Let AB be divided at C.

Produce AB to D, making BD equal to BC.

Then shall four times the rect. AB, BC *with the sq. on* AC = *the sq. on* AD.

On AD describe the square AEFD; and complete the construction as indicated in the figure.

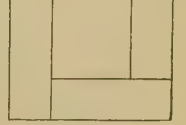
Euclid then proves (i) that the figs. CK, BN, GR, KO are all equal:

(ii) that the figs. AG, MP, PL, RF are all equal.

Hence the eight figures named above are together four times the sum of the figs. AG, CK; that is, four times the fig. AK; that is, four times the rect. AB, BC.

But the whole fig. AF, namely the sq. on AD, is made up of these eight figures, together with the fig. XH, which is the sq. on AC:

hence the sq. on AD = four times the rect. AB, BC, together with the sq. on AC. Q.E.D.



The accompanying figure will suggest a less cumbrous proof, which we leave as an Exercise to the student.

CORRESPONDING ALGEBRAICAL FORMULA.

The result of this proposition may be written

$$4\,AB \,.\, BC + AC^2 = AD^2.$$

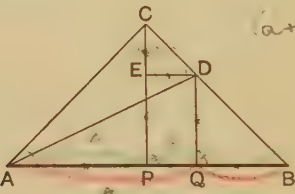
Let $AB = a$, and $BC = b$; then $AC = a - b$, and $AD = a + b$.

Hence we have $4ab + (a-b)^2 = (a+b)^2$;

or $$(a+b)^2 - (a-b)^2 = 4ab.$$

PROPOSITION 9. THEOREM. [EUCLID'S PROOF.]

If a straight line is divided equally and also unequally, the sum of the squares on the two unequal parts is twice the sum of the squares on half the line and on the line between the points of section.



Let the straight line AB be divided equally at P, and unequally at Q.

Then shall the sum of the sqq. on AQ, QB *be twice the sum of the sqq. on* AP, PQ.

Construction. At P draw PC at rt. angles to AB; I. 11.
and make PC equal to AP or PB. I. 3.
Join AC, BC.
Through Q draw QD par^{l} to PC; I. 31.
and through D draw DE par^{l} to AB.
Join AD.

Proof. Then since PA = PC, *Constr.*
∴ the angle PAC = the angle PCA. I. 5.
And since, in the triangle APC, the angle APC is a rt. angle, *Constr.*
∴ the sum of the angles PAC, PCA is a rt. angle: I. 32.
hence each of the angles PAC, PCA is half a rt. angle.
So also, each of the angles PBC, PCB is half a rt. angle.
∴ the whole angle ACB is a rt. angle.
Again, the ext. angle CED = the int. opp. angle CPB; I. 29.
∴ the angle CED is a rt. angle:
and the angle ECD is half a rt. angle. *Proved.*
∴ the remaining angle EDC is half a rt. angle; I. 32.
∴ the angle ECD = the angle EDC;
∴ EC = ED. I. 6.

Again, the ext. angle DQB = the int. opp. angle CPB ; I. 29.
∴ the angle DQB is a rt. angle.
And the angle QBD is half a rt. angle ; *Proved.*
∴ the remaining angle QDB is half a rt. angle ; I. 32.
∴ the angle QBD = the angle QDB ;
∴ QD = QB.

Now the sq. on AP = the sq. on PC ; for AP = PC. *Constr.*
And since the angle APC is a rt. angle,
∴ the sq. on AC = the sum of the sqq. on AP, PC ; I. 47.
∴ the sq. on AC is twice the sq. on AP.

Similarly, the sq. on CD is twice the sq. on ED, that is, twice the sq. on the opp. side PQ. I. 34.

Now the sqq. on AQ, QB = the sqq. on AQ, QD *Proved.*
= the sq. on AD, for AQD is a rt. angle ; I. 47.
= the sum of the sqq. on AC, CD, for ACD is a rt. angle ; I. 47.
= twice the sq. on AP with twice the sq. on PQ. *Proved.*

That is,
the sum of the sqq. on AQ, QB = twice the sum of the sqq. on AP, PQ. Q.E.D.

CORRESPONDING ALGEBRAICAL FORMULA.

The result of this proposition may be written

$$AQ^2 + QB^2 = 2(AP^2 + PQ^2).$$

Let $AB = 2a$; and $PQ = b$;

then AP and PB each $= a$.

Also $AQ = a + b$; and $QB = a - b$.

Hence the statement

$$AQ^2 + QB^2 = 2(AP^2 + QP^2)$$

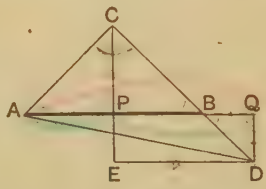
becomes

$$(a+b)^2 + (a-b)^2 = 2(a^2 + b^2).$$

[NOTE. For alternative proofs of this proposition, see page 148.]

PROPOSITION 10. THEOREM. [EUCLID'S PROOF.]

If a straight line is bisected and produced to any point, the sum of the squares on the whole line thus produced, and on the part produced, is twice the sum of the squares on half the line bisected and on the line made up of the half and the part produced.



Let the st. line AB be bisected at P, and produced to Q.
Then shall the sum of the sqq. on AQ, QB *be twice the sum of the sqq. on* AP, PQ.

Construction. At P draw PC at right angles to AB ; I. 11.
and make PC equal to PA or PB. I. 3.
Join AC, BC.
Through Q draw QD par^{l} to PC, to meet CB produced in D ; I. 31.
and through D draw DE par^{l} to AB, to meet CP produced in E.
Join AD.

Proof. Then since PA = PC, *Constr.*
∴ the angle PAC = the angle PCA. I. 5.
And since, in the triangle APC, the angle APC is a rt. angle,
∴ the sum of the angles PAC, PCA is a rt. angle. I. 32.
Hence each of the angles PAC, PCA is half a rt. angle.
So also, each of the angles PBC, PCB is half a rt. angle.
∴ the whole angle ACB is a rt. angle.

Again, the ext. angle CPB = the int. opp. angle CED : I. 29.
∴ the angle CED is a rt. angle :
and the angle ECD is half a rt. angle ; *Proved.*
∴ the remaining angle EDC is half a rt. angle. I. 32.
∴ the angle ECD = the angle EDC ;
∴ EC = ED. I. 6.

Again, the angle DQB = the alt. angle CPB ; I. 29.
∴ the angle DQB is a rt. angle.
Also the angle QBD = the vert. opp. angle CBP : I. 15.
that is, the angle QBD is half a rt. angle.
∴ the remaining angle QDB is half a rt. angle : I. 32.
∴ the angle QBD = the angle QDB ;
∴ QB = QD. I. 6.

Now the sq. on AP = the sq. on PC ; for AP = PC. *Constr.*
And since the angle APC is a rt. angle,
∴ the sq. on AC = the sum of the sqq. on AP, PC ; I. 47.
∴ the sq. on AC is twice the sq. on AP.

Similarly, the sq. on CD is twice the sq. on ED, that is, twice the sq. on the opp. side PQ. I. 34.

Now the sqq. on AQ, QB = the sqq. on AQ, QD *Proved.*
= the sq. on AD, for AQD is a rt. angle ; I. 47.
= the sum of the sqq. on AC, CD, for ACD is a rt. angle ; I. 47.
= twice the sq. on AP with twice the sq. on PQ. *Proved.*

That is,
the sum of the sqq. on AQ, QB is twice the sum of the sqq. on AP, PQ. Q.E.D.

CORRESPONDING ALGEBRAICAL FORMULA.

The result of this proposition may be written

$$AQ^2 + QB^2 = 2(AP^2 + PQ^2).$$

Let $AB = 2a$; and $PQ = b$;
then AP and PB each $= a$.
Also $AQ = a + b$; and $QB = b - a$.

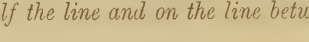
Hence we have

$$(a+b)^2 + (b-a)^2 = 2(a^2 + b^2).$$

[NOTE. For alternative proofs of this proposition, see page 149.]

Proposition 9. [Alternative Proof.]

If a straight line is divided equally and also unequally, the sum of the squares on the two unequal parts is twice the sum of the squares on half the line and on the line between the points of section.



Let the straight line AB be divided equally at P and unequally at Q.

Then shall the sum of the sqq. on AQ, QB *be twice the sum of the sqq. on* AP, PQ.

Proof.

The sq. on AQ = the sum of the sqq. on AP, PQ with twice the rect. AP, PQ II. 4.
= the sum of the sqq. on AP, PQ with twice the rect. PB, PQ; for PB = AP.

To each of these equals add the sq. on QB.

Then the sqq. on AQ, QB = the sum of the sqq. on AP, PQ with twice the rect. PB, PQ and the sq. on QB.

But twice the rect. PB, PQ and the sq. on QB
= the sum of the sqq. on PB, PQ. II. 7.

∴ the sqq. on AQ, QB = the sum of the sqq. on AP, PQ with the sum of the sqq. on PB, PQ
= *twice* the sum of the sqq. on AP, PQ.

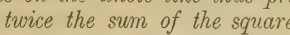
Q.E.D.

Note. The following concise proof, obtained from II. 4 and II. 5, is useful as an exercise, but it is hardly admissible as a formal demonstration owing to its algebraical use of the negative sign.

We have $AQ^2 + QB^2 = AB^2 - 2AQ \cdot QB$ II. 4.
$= 4PB^2 - 2AQ \cdot QB$ II. 4, *Cor.* 2.
$= 4PB^2 - 2(PB^2 - PQ^2)$ II. 5.
$= 2PB^2 + 2PQ^2$.

PROPOSITION 10. [ALTERNATIVE PROOF.]

If a straight line is bisected and produced to any point, the sum of the squares on the whole line thus produced and on the part produced, is twice the sum of the squares on half the line bisected and on the line made up of the half and the part produced.



Let the straight line AB be bisected at P, and produced to Q.

Then shall the sum of the sqq. on AQ, QB *be twice the sum of the sqq. on* AP, PQ.

Proof.

The sq. on AQ = the sum of the sqq. on AP, PQ with twice the rect. AP, PQ II. 4.
= the sum of the sqq. on AP, PQ with twice the rect. PB, PQ; for PB = AP.

To each of these equals add the sq. on QB.

Then the sqq. on AQ, QB = the sum of the sqq. on AP, PQ with twice the rect. PB, PQ and the sq. on QB.

But twice the rect. PB, PQ and the sq. on QB
= the sum of the sqq. on PB, PQ. II. 7.

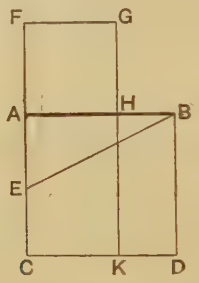
∴ the sqq. on AQ, QB = the sum of the sqq. on AP, PQ with the sum of the sqq. on PB, PQ
= *twice* the sum of the sqq. on AP, PQ.
Q.E.D.

NOTE. Another proof of this proposition, based on II. 7 and II. 6, is indicated by the following steps:

We have $AQ^2 + QB^2 = 2AQ \cdot QB + AB^2$ II. 7.
$= 2AQ \cdot QB + 4PB^2$ II. 4, *Cor.* 2.
$= 2(PQ^2 - PB^2) + 4PB^2$ II. 6.
$= 2PB^2 + 2PQ^2$.

Proposition 11. Problem.

To divide a given straight line into two parts, so that the rectangle contained by the whole and one part may be equal to the square on the other part.



Let AB be the given straight line.

It is required to divide AB *into two parts, so that the rectangle contained by the whole and one part may be equal to the square on the other part.*

Construction. On AB describe the square ACDB. I. 46.
Bisect AC at E. I. 10.
Join EB.
Produce CA to F, making EF equal to EB. I. 3.
On AF describe the square AFGH. I. 46.

Then shall AB *be divided at* H, *so that the rect.* AB, BH *is equal to the sq. on* AH.

Produce GH to meet CD in K.

Proof. Because CA is bisected at E, and produced to F,
∴ the rect. CF, FA with the sq. on EA = the sq. on EF II. 6.
= the sq. on EB. *Constr.*
But the sq. on EB = the sum of the sqq. on EA, AB,
for the angle EAB is a rt. angle. I. 47.
∴ the rect. CF, FA with the sq. on EA = the sum of the sqq. on EA, AB.
From these equals take the sq. on EA:
then the rect. CF, FA = the sq. on AB.

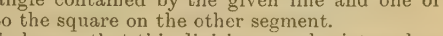
But the rect. CF, FA = the fig. FK; for FA = FG;
and the sq. on AB = the fig. AD. *Constr.*
∴ the fig. FK = the fig. AD.
From these equals take the common fig. AK;
then the remaining fig. FH = the remaining fig. HD.
But the fig. HD = the rect. AB, BH; for BD = AB;
and the fig. FH is the sq. on AH.
∴ the rect. AB, BH = the sq. on AH. Q.E.F.

DEFINITION. A straight line is said to be divided in **Medial Section** when the rectangle contained by the given line and one of its segments is equal to the square on the other segment.

The student should observe that this division may be *internal* or *external*.

Thus if the straight line AB is divided internally at H, and externally at H′, so that

(i) $AB.BH = AH^2$,
(ii) $AB.BH' = AH'^2$,



we shall in either case consider that AB is divided in medial section.

The case of *internal* section is alone given in Euclid II. 11; but a straight line may be divided *externally* in medial section by a similar process. See Ex. 21, p. 160.

ALGEBRAICAL ILLUSTRATION.

It is required to find a point H in AB, or AB produced, such that

$$AB.BH = AH^2.$$

Let AB contain a units of length, and let AH contain x units;

then $BH = a - x$:

and x must be such that $a(a-x) = x^2$,

or $$x^2 + ax - a^2 = 0.$$

Thus the construction for dividing a straight line in medial section corresponds to the solution of this quadratic equation, the two roots of which indicate the *internal* and *external* points of division.

EXERCISES.

In the figure of II. 11, shew that

(i) if CH is produced to meet BF at L, CL is at right angles to BF;
(ii) if BE and CH meet at O, AO is at right angles to CH.
(iii) the lines BG, DF, AK are parallel:
(iv) CF is divided in medial section at A.

Proposition 12. Theorem.

In an obtuse-angled triangle, if a perpendicular is drawn from either of the acute angles to the opposite side produced, the square on the side subtending the obtuse angle is greater than the sum of the squares on the sides containing the obtuse angle, by twice the rectangle contained by the side on which, when produced, the perpendicular falls, and the line intercepted without the triangle, between the perpendicular and the obtuse angle.

Let ABC be an obtuse-angled triangle, having the obtuse angle at C; and let AD be drawn from A perp. to the opp. side BC produced.

Then shall the sq. on AB *be greater than the sum of the sqq. on* BC, CA, *by twice the rect.* BC, CD.

Proof. Because BD is divided into two parts at C,
∴ the sq. on BD = the sum of the sqq. on BC, CD, with twice the rect. BC, CD. II. 4,

To each of these equals add the sq. on DA.

Then the sqq. on BD, DA = the sum of the sqq. on BC, CD, DA, with twice the rect. BC, CD.

But the sum of the sqq. on BD, DA = the sq. on AB, for the angle at D is a rt. angle. I. 47.

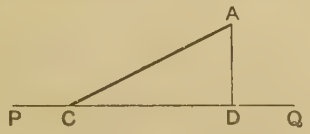
Similarly, the sum of the sqq. on CD, DA = the sq. on CA.

∴ the sq. on AB = the sum of the sqq. on BC, CA, with twice the rect. BC, CD.

That is, the sq. on AB is greater than the sum of the sqq. on BC, CA by twice the rect. BC, CD. Q.E.D.

NOTE ON PROP. 12.

A general definition of the **projection of one straight line on another** is given on page 105. The student's attention is here called to a special case of projection which will enable us to simplify the Enunciation of Proposition 12.



In the above diagram, CA is a given straight line drawn from a point C in PQ; and from A a perpendicular AD is drawn to PQ. In this case, CD is said to be the **projection** of CA on PQ.

By applying this definition to the figure of Prop. 12, we see that the statement

The sq. on AB *is greater than the sum of the sqq. on* BC, CA *by twice the rect.* BC, CD

is the particular form of the following general Enunciation:

In an obtuse-angled triangle the square on the side opposite the obtuse angle is greater than the sum of the squares on the sides containing the obtuse angle by twice the rectangle contained by one of those sides, and the **projection of the other side** *upon it.*

The Enunciation of Prop. 12 thus stated should be carefully compared with that of Prop. 13.

Proposition 13. Theorem.

In every triangle, the square on the side subtending an acute angle is less than the sum of the squares on the sides containing that angle, by twice the rectangle contained by either of these sides, and the straight line intercepted between the perpendicular let fall on it from the opposite angle, and the acute angle.

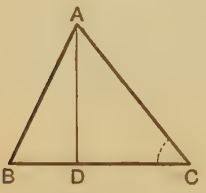


Fig. 1.

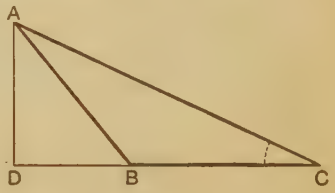


Fig. 2.

Let ABC be any triangle having the angle at C an acute angle; and let AD be the perp. drawn from A to the opp. side BC.

Then shall the sq. on AB *be less than the sum of the sqq. on* BC, CA, *by twice the rect.* BC, CD.

Proof. Now AD may fall within the triangle ABC, as in fig. 1, or without it, as in fig. 2.

Because { in fig. 1, BC is divided into two parts at D,
in fig. 2, DC is divided into two parts at B,

$\therefore$ *in both cases*

the sum of the sqq. on BC, CD = twice the rect. BC, CD with the sq. on BD. II. 7.

To each of these equals add the sq. on DA.

Then the sum of the sqq. on BC, CD, DA = twice the rect. BC, CD with the sum of the sqq. on BD, DA.

But the sum of the sqq. on CD, DA = the sq. on CA, I. 47. for the angle ADC is a rt. angle.

Similarly, the sum of the sqq. on BD, DA = the sq. on AB.

$\therefore$ the sum of the sqq. on BC, CA = twice the rect. BC, CD with the sq. on AB.

That is, the sq. on AB is less than the sum of the sqq. on BC, CA by twice the rect. BC, CD. Q.E.D.

Obs. If the perpendicular AD *coincides* with AB, that is, if ABC is a right angle, then *twice the rect.* BC, CD becomes *twice the sq. on* BC; and it may be shewn that the proposition merely repeats the result of I. 47.

NOTES ON PROP. 13.

(i) Remembering the definition of the **projection** of a straight line given on p. 153, we may enunciate Prop. 13 as follows;

In every triangle, the square on the side subtending an acute angle is less than the sum of the squares on the sides containing that angle, by twice the rectangle contained by one of these sides and the **projection of the other side** *upon it.*

(ii) Comparing the Enunciations of II. 12, I. 47, II. 13, we see that in the triangle ABC,

if the angle ACB is *obtuse*, we have by II. 12,

$$AB^2 = BC^2 + CA^2 + 2BC \cdot CD;$$

if the angle ACB is *a right angle*, we have by I. 47,

$$AB^2 = BC^2 + CA^2;$$

if the angle ACB is *acute*, we have by II. 13,

$$AB^2 = BC^2 + CA^2 - 2BC \cdot CD.$$

These results may be collected as follows:

The square on a side of a triangle is greater than, equal to, or less than the sum of the squares on the other sides, according as the angle opposite to the first is obtuse, a right angle, or acute.

EXERCISES ON II. 12 AND 13.

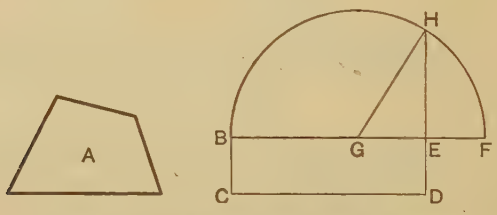
1. If from one of the base angles of an isosceles triangle a perpendicular is drawn to the opposite side, then twice the rectangle contained by that side and the segment adjacent to the base is equal to the square on the base.

2. If one angle of a triangle is one-third of two right angles, shew that the square on the opposite side is less than the sum of the squares on the sides forming that angle, by the rectangle contained by these two sides. [See Ex. 10, p. 109.]

3. If one angle of a triangle is two-thirds of two right angles, shew that the square on the opposite side is greater than the sum of the squares on the sides forming that angle, by the rectangle contained by these sides. [See Ex. 10, p. 109.]

Proposition 14. Problem.

To describe a square that shall be equal to a given rectilineal figure.



Let A be the given rectilineal figure.
It is required to describe a square equal to A.

Construction. Describe a par^{m} BCDE equal to the fig. A, and having the angle CBE a right angle. I. 45.
Then if BC = BE, the fig BD is a square; and what was required is done.
But if not, produce BE to F, making EF equal to ED, I. 3.
and bisect BF at G. I. 10.
With centre G, and radius GF, describe the semicircle BHF; produce DE to meet the semicircle at H.
Then shall the sq. on EH *be equal to the given fig.* A.
Join GH.

Proof. Because BF is divided equally at G and unequally at E,
∴ the rect. BE, EF with the sq. on GE = the sq. on GF II. 5.
= the sq. on GH.
But the sq. on GH = the sum of the sqq. on GE, EH;
for the angle HEG is a rt. angle. I. 47.
∴ the rect. BE, EF with the sq. on GE = the sum of the sqq. on GE, EH.
From these equals take the sq. on GE:
then the rect. BE, EF = the sq. on HE.
But the rect. BE, EF = the fig. BD; for EF = ED; *Constr.*
and the fig. BD = the given fig. A. *Constr.*
∴ the sq. on EH = the given fig. A. Q.E.F.

QUESTIONS FOR REVISION ON BOOK II.

1. Explain the phrase, *the rectangle contained by* AB, CD; and shew by superposition that if AB = PQ, and CD = RS, then the rectangle contained by AB, CD = the rectangle contained by PQ, RS.

2. Shew that Prop. 2 is a *special case* of Prop. 1, explaining under what conditions Prop. 1 becomes identical with Prop. 2.

3. What must be the relation between the divided and undivided lines in the enunciation of Prop. 1 in order to give the result proved in Prop. 3?

4. Define the *segments into which a straight line is divided at a point* in such a way as to be applicable to the case when the dividing point is in the given line produced.

Hence frame a statement which includes the enunciations of both II. 5 and II. 6, and find the algebraical formulae corresponding to these enunciations.

Also combine in a single enunciation the results of I. 9 and II. 10.

5. Compare the results proved in Propositions 4 and 7 by finding the algebraical formulae corresponding to their enunciations.

6. *The difference of the squares on two straight lines is equal to the rectangle contained by their sum and difference.* Deduce this theorem from Prop. 5.

7. Define *the projection of one straight line on another.*

How may the enunciations of II. 12 and II. 13 be simplified by means of this definition?

8. In the figure of Proposition 14,

(i) If BE = 8 inches, and ED = 2 inches, find the length of EH.

(ii) If BE = 12·5 inches, and EH = 2·5 inches, find the length of ED.

(iii) If BE = 9 inches, and EH = 3 inches, find the length of GH.

9. When is a straight line said to be divided in *medial section*?

If a straight line 8 inches in length is divided internally in medial section, shew that the lengths of the segments are approximately 4·9 inches and 3·1 inches.

[Frame a quadratic equation as explained on page 151, and solve.]

THEOREMS AND EXAMPLES ON BOOK II.

ON II. 4 AND 7.

1. *Shew by* II. 4 *that the square on a straight line is four times the square on half the line.*

[This result is constantly used in solving examples on Book II., especially those which follow from II. 12 and 13.]

2. If a straight line is divided into any three parts, the square on the whole line is equal to the sum of the squares on the three parts together with twice the rectangles contained by each pair of these parts.

Shew that the algebraical formula corresponding to this theorem is

$$(a+b+c)^2=a^2+b^2+c^2+2bc+2ca+2ab.$$

3. *In a right-angled triangle, if a perpendicular is drawn from the right angle to the hypotenuse, the square on this perpendicular is equal to the rectangle contained by the segments of the hypotenuse.*

4. In an isosceles triangle, if a perpendicular is drawn from one of the angles at the base to the opposite side, shew that the square on the perpendicular is equal to twice the rectangle contained by the segments of that side together with the square on the segment adjacent to the base.

5. Any rectangle is half the rectangle contained by the diagonals of the squares described upon its two sides.

6. In any triangle if a perpendicular is drawn from the vertical angle to the base, the sum of the squares on the sides forming that angle, together with twice the rectangle contained by the segments of the base, is equal to the square on the base together with twice the square on the perpendicular.

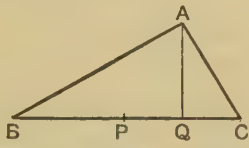
ON II. 5 AND 6.

Obs. The student is reminded that these important propositions are both included in the following enunciation :

The difference of the squares on two straight lines is equal to the rectangle contained by their sum and difference. [See Cor., p. 137].

7. In a right-angled triangle the square on one of the sides forming the right angle is equal to the rectangle contained by the sum and difference of the hypotenuse and the other side. [I. 47 and II. 5, Cor.]

8. *The difference of the squares on two sides of a triangle is equal to twice the rectangle contained by the base and the intercept between the middle point of the base and the foot of the perpendicular drawn from the vertical angle to the base.*



Let ABC be a triangle, and let P be the middle point of the base BC: let AQ be drawn perp. to BC.

Then shall $AB^2 - AC^2 = 2BC \cdot PQ$.

First, let AQ fall within the triangle.

Now $AB^2 = BQ^2 + QA^2$, I. 47.

also $AC^2 = QC^2 + QA^2$,

$\therefore\ AB^2 - AC^2 = BQ^2 - QC^2$ Ax. 3.

$= (BQ + QC)(BQ - QC)$ II. 5, Cor.

$= BC \cdot 2PQ$ Ex., p. 137.

$= 2BC \cdot PQ$ Q.E.D.

The case in which AQ falls outside the triangle presents no difficulty.

9. *The square on any straight line drawn from the vertex of an isosceles triangle to the base is less than the square on one of the equal sides by the rectangle contained by the segments of the base.*

10. The square on any straight line drawn from the vertex of an isosceles triangle to the base produced, is greater than the square on one of the equal sides by the rectangle contained by the *segments into which the base is divided externally.*

11. If a straight line is drawn through one of the angles of an equilateral triangle to meet the opposite side produced, so that the rectangle contained *by the segments of the base* is equal to the square on the side of the triangle; shew that the square on the line so drawn is double of the square on a side of the triangle.

12. If XY is drawn parallel to the base BC of an isosceles triangle ABC, then the difference of the squares on BY and CY is equal to the rectangle contained by BC, XY. [See above, Ex. 8.]

13. In a right-angled triangle, if a perpendicular is drawn from the right angle to the hypotenuse, the square on either side forming the right angle is equal to the rectangle contained by the hypotenuse and the segment of it adjacent to that side.

ON II. 9 AND 10.

14. Deduce Prop. 9 from Props. 4 and 5, using also the theorem that the square on a straight line is four times the square on half the line.

15. Deduce Prop. 10 from Props. 7 and 6, using also the theorem mentioned in the preceding Exercise.

16. If a straight line is divided equally, and also unequally, the squares on the two unequal segments are together equal to twice the rectangle contained by these segments together with four times the square on the line between the points of section.

ON II. 11.

17. *If a straight line is divided internally in medial section, and from the greater segment a part be taken equal to the less, shew that the greater segment is also divided in medial section.*

18. If a straight line is divided in medial section, the rectangle contained by the sum and difference of the segments is equal to the rectangle contained by the segments.

19. If AB is divided at H in medial section, and if X is the middle point of the greater segment AH, shew that a triangle whose sides are equal to AH, XH, BX respectively must be right-angled.

20. If a straight line AB is divided internally in medial section at H, prove that the sum of the squares on AB, BH is three times the square on AH.

21. *Divide a straight line externally in medial section.*

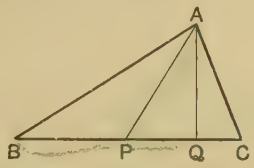
[Proceed as in II. 11, but instead of drawing EF, make EF′ equal to EB in the direction remote from A; and on AF′ describe the square AF′G′H′ on the side remote from AB. Then AB will be divided externally at H′ as required.]

ON II. 12 AND 13.

22. In a triangle ABC the angles at B and C are acute: if E and F are the feet of perpendiculars drawn from the opposite angles to the sides AC, AB, shew that the square on BC is equal to the sum of the rectangles AB, BF and AC, CE.

23. ABC is a triangle right-angled at C, and DE is drawn from a point D in AC perpendicular to AB: shew that the rectangle AB, AE is equal to the rectangle AC, AD.

24. *In any triangle the sum of the squares on two sides is equal to twice the square on half the third side together with twice the square on the median which bisects the third side.*



Let ABC be a triangle, and AP the median bisecting the side BC.
Then shall $AB^2+AC^2=2BP^2+2AP^2$.

Draw AQ perp. to BC.

Consider the case in which AQ falls within the triangle, but does not coincide with AP.

Now of the angles APB, APC, one must be obtuse, and the other acute: let APB be obtuse.

Then in the $\triangle$ APB, $AB^2=BP^2+AP^2+2BP\,.\,PQ$, II. 12.

Also in the $\triangle$ APC, $AC^2=CP^2+AP^2-2CP\,.\,PQ$. II. 13.

But CP = BP,

$\therefore$ $CP^2=BP^2$; and the rect. BP, PQ = the rect. CP, PQ,

Hence adding the above results,

$AB^2+AC^2=2\,.\,BP^2+2\,.\,AP^2$. Q.E.D.

The student will have no difficulty in adapting this proof to the cases in which AQ falls without the triangle, or coincides with AP.

25. *The sum of the squares on the sides of a parallelogram is equal to the sum of the squares on the diagonals.*

26. In any quadrilateral the squares on the diagonals are together equal to twice the sum of the squares on the straight lines joining the middle points of opposite sides. [See Ex. 9, p. 105.]

27. If from any point within a rectangle straight lines are drawn to the angular points, the sum of the squares on one pair of the lines drawn to opposite angles is equal to the sum of the squares on the other pair.

28. The sum of the squares on the sides of a quadrilateral is greater than the sum of the squares on its diagonals by four times the square on the straight line which joins the middle points of the diagonals.

29. O is the middle point of a given straight line AB, and from O as centre, any circle is described: if P be any point on its circumference, shew that the sum of the squares on AP, BP *is constant.*

30. Given the base of a triangle, and the sum of the squares on the sides forming the vertical angle ; find the locus of the vertex.

31. ABC is an isosceles triangle in which AB and AC are equal. AB is produced beyond the base to D, so that BD is equal to AB. Shew that the square on CD is equal to the square on AB together with twice the square on BC.

32. In a right-angled triangle the sum of the squares on the straight lines drawn from the right angle to the points of trisection of the hypotenuse is equal to five times the square on the line between the points of trisection.

33. Three times the sum of the squares on the sides of a triangle is equal to four times the sum of the squares on the medians.

34. ABC is a triangle, and O the point of intersection of its medians : shew that

$$AB^2+BC^2+CA^2=3(OA^2+OB^2+OC^2).$$

35. ABCD is a quadrilateral, and X the middle point of the straight line joining the bisections of the diagonals ; with X as centre *any* circle is described, and P is any point upon this circle : shew that $PA^2+PB^2+PC^2+PD^2$ is *constant*, being equal to

$$XA^2+XB^2+XC^2+XD^2+4XP^2.$$

36. The squares on the diagonals of a trapezium are together equal to the sum of the squares on its two oblique sides, with twice the rectangle contained by its parallel sides.

PROBLEMS.

37. Construct a rectangle equal to the difference of two squares.

38. Divide a given straight line into two parts so that the rectangle contained by them may be equal to the square described on a given straight line which is less than half the straight line to be divided.

39. Given a square and one side of a rectangle which is equal to the square, find the other side.

40. Produce a given straight line so that the rectangle contained by the whole line thus produced and the part produced, may be equal to the square on another given line.

41. Produce a given straight line so that the rectangle contained by the whole line thus produced and the given line shall be equal to the square on the part produced.

42. Divide a straight line AB into two parts at C, such that the rectangle contained by BC and another line X may be equal to the square on AC.

BOOK III.

Book III. deals with the properties of Circles.

For convenience of reference the following definitions are repeated from Book I.

I. *Def.* 15. A **circle** is a plane figure bounded by one line, which is called the **circumference**, and is such that all straight lines drawn from a certain point within the figure to the circumference are equal to one another: this point is called the **centre** of the circle.

NOTE. Circles which have the same centre are said to be **concentric.**

I. *Def.* 16. A **radius** of a circle is a straight line drawn from the centre to the circumference.

I. *Def.* 17. A **diameter** of a circle is a straight line drawn through the centre, and terminated both ways by the circumference.

I. *Def.* 18. A **semicircle** is the figure bounded by a diameter of a circle and the part of the circumference cut off by the diameter.

NOTE. From these definitions we draw the following inferences:

(i) The distance of a point from the centre of a circle is less than the radius, if the point is within the circumference: and the distance of a point from the centre is greater than the radius, if the point is without the circumference.

(ii) A point is within a circle if its distance from the centre is less than the radius: and a point is without a circle if its distance from the centre is greater than the radius.

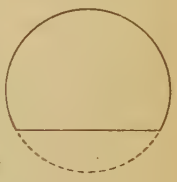
(iii) Circles of equal radius are equal in all respects; that is to say, their areas and circumferences are equal.

(iv) A circle is divided by any diameter into two parts which are equal in all respects.

Definitions to Book III.

1. An **arc** of a circle is any part of the circumference.

2. A **chord** of a circle is the straight line which joins any two points on the circumference.

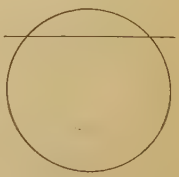


Note. From these definitions it may be seen that a chord of a circle, which does not pass through the centre, divides the circumference into two unequal arcs; of these, the greater is called the **major arc**, and the less the **minor arc**. Thus the major arc is *greater*, and the minor arc *less* than the semi-circumference.

The major and minor arcs, into which a circumference is divided by a chord, are said to be **conjugate** to one another.

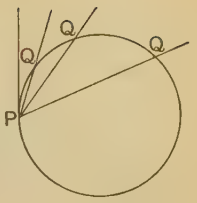
3. Chords of a circle are said to be **equidistant** from the centre, when the perpendiculars drawn to them from the centre are equal:
and one chord is said to be **further from the centre** than another, when the perpendicular drawn to it from the centre is greater than the perpendicular drawn to the other.

4. A **secant** of a circle is a straight line of indefinite length, which cuts the circumference in two points.



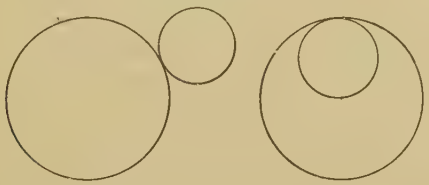
5. A **tangent** to a circle is a straight line which meets the circumference, but being produced, does not cut it. Such a line is said to **touch** the circle at a point; and the point is called the **point of contact.**

NOTE. If a secant, which cuts a circle at the points P and Q, gradually changes its position in such a way that P remains fixed, the point Q will ultimately approach the fixed point P, until at length these points may be made to coincide. When the straight line PQ reaches this limiting position, it becomes the *tangent* to the circle at the point P.



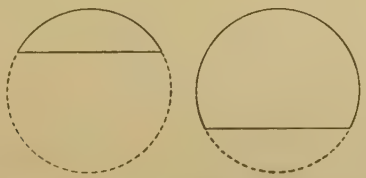
Hence a tangent may be defined as a straight line which passes through *two coincident points* on the circumference.

6. Circles are said to **touch one another** when they meet, but do not cut one another.



NOTE. When each of the circles which meet is *outside the other*, they are said to touch one another **externally**, or to have **external contact**: when one of the circles is *within the other*, the first is said to touch the other **internally**, or to have **internal contact** with it.

7. A **segment** of a circle is the figure bounded by a chord and one of the two arcs into which the chord divides the circumference.



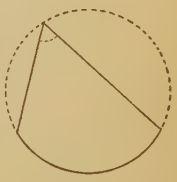
NOTE. The chord of a segment is sometimes called its base.

8. An **angle in a segment** is one formed by two straight lines drawn from any point in the arc of the segment to the extremities of its chord.

NOTE. (i) It will be shewn in Proposition 21, that all angles in the same segment of a circle are equal.

NOTE. (ii) The angle *of* a segment (as distinct from the angle *in* a segment) is sometimes defined as that which is contained between the *chord* and the *arc*; but this definition is not required in any proposition of Euclid.

9. An **angle at the circumference** of a circle is one formed by straight lines drawn from a point on the circumference to the extremities of an arc: such an angle is said to **stand upon** the arc by which it is subtended.



10. **Similar segments** of circles are those which contain equal angles.

11. A **sector** of a circle is a figure bounded by two radii and the arc intercepted between them.

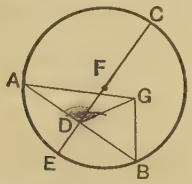
SYMBOLS AND ABBREVIATIONS.

In addition to the symbols and abbreviations given on page 11, we shall use the following.

$\odot$ *for* circle, $\bigcirc^{ce}$ *for* circumference.

PROPOSITION 1. PROBLEM.

To find the centre of a given circle.



Let ABC be a given circle.

It is required to find the centre of the ⊙ ABC.

Construction. In the given circle draw any chord AB, and bisect AB at D. I. 10.

From D draw DC at right angles to AB; I. 11.
and produce DC to meet the ○ce at E and C.

Bisect EC at F. I. 10.

Then shall F *be the centre of the* ⊙ ABC.

Proof. First, the centre of the circle must be in EC: for if not, suppose the centre to be at a point G outside EC.

Join AG, DG, BG.

Then in the △s GDA, GDB,

Because { DA = DB, *Constr.*
and GD is common;
and GA = GB, for by supposition they are radii;

∴ the ∠ GDA = the ∠ GDB; I. 8.

∴ these angles, being adjacent, are rt. angles.

But the ∠ CDB is a rt. angle; *Constr.*

∴ the ∠ GDB = the ∠ CDB, *Ax.* 11.

the part equal to the whole, which is impossible.

∴ G is not the centre.

So it may be shewn that no point outside EC is the centre;

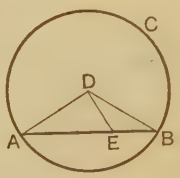
∴ the centre lies in EC.

∴ F, the middle point of the diameter EC, must be the centre of the ⊙ ABC. Q.E.F.

COROLLARY. *The straight line which bisects a chord of a circle at right angles passes through the centre.*

Proposition 2. Theorem.

If any two points are taken in the circumference of a circle, the chord which joins them falls within the circle.



Let ABC be a circle, and A and B any two points on its ○ce.

Then shall the chord AB *fall within the circle.*

Construction. Find D, the centre of the ⊙ ABC; III. 1.
and in AB take any point E.
Join DA, DE, DB.

Proof. In the △ DAB, because DA = DB, I. *Def.* 15.
∴ the ∠ DAB = the ∠ DBA. I. 5.

But the ext. ∠ DEB is greater than the int. opp. ∠ DAE; I. 16.

∴ the ∠ DEB is also greater than the ∠ DBE.

∴ in the △ DEB, the side DB, which is opposite the greater angle, is greater than DE which is opposite the less: I. 19.

that is to say, DE is less than DB, a radius of the circle;

∴ E falls within the circle.

Similarly, any other point between A and B may be shewn to fall within the circle.

∴ AB falls within the circle. Q.E.D.

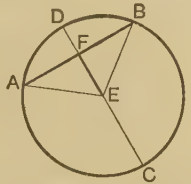
Note. A part of a curved line is said to be **concave** to a point, when for every chord (taken so as to lie between the point and the curve) all straight lines drawn from the given point to the intercepted arc are cut by the chord: if, when any chord whatever is taken, no straight line drawn from the given point to the intercepted arc is cut by the chord, the curve is said to be **convex** to that point.

Proposition 2 proves that the whole circumference of a circle is *concave to its centre.*

PROPOSITION 3. THEOREM.

If a straight line drawn through the centre of a circle bisects a chord which does not pass through the centre, it shall cut the chord at right angles.

Conversely, if it cuts the chord at right angles, it shall bisect it.



Let ABC be a circle; and let CD be a st. line drawn through the centre, and AB a chord which does not pass through the centre.

First. Let CD bisect the chord AB at F.
Then shall CD *cut* AB *at rt. angles.*

Construction. Find E the centre of the circle; and join EA, EB. III. 1.

Proof. Then in the $\triangle^s$ AFE, BFE,

Because { AF = BF, *Hyp.*
and FE is common;
and AE = BE, being radii of the circle;

$\therefore$ the $\angle$ AFE = the $\angle$ BFE; I. 8.

$\therefore$ these angles, being adjacent, are rt. angles;
that is, DC cuts AB at rt. angles. Q.E.D.

Conversely. Let CD cut the chord AB at rt. angles.
Then shall CD *bisect* AB *at* F.

Construction. Find E the centre; and join EA, EB.

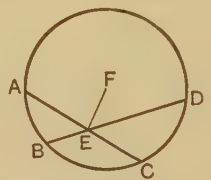
Proof. In the $\triangle$ EAB, because EA = EB, I. *Def.* 15.
$\therefore$ the $\angle$ EAB = the $\angle$ EBA. I. 5.
Then in the $\triangle^s$ EFA, EFB,

Because { the $\angle$ EAF = the $\angle$ EBF, *Proved.*
and the $\angle$ EFA = the $\angle$ EFB, being rt. angles; *Hyp.*
and EF is common;

$\therefore$ AF = BF; I. 26.
that is, CD bisects AB at F. Q.E.D.

Proposition 4. Theorem.

If in a circle two chords cut one another, which do not both pass through the centre, they cannot both be bisected at their point of intersection.



Let ABCD be a circle, and AC, BD two chords which intersect at E, but do not both pass through the centre.

Then AC *and* BD *shall not be* both *bisected at* E.

Case I. If *one* chord passes through the centre, it is a diameter, and the centre is its middle point;

∴ it cannot be bisected by the other chord, which by hypothesis does not pass through the centre.

Case II. If neither chord passes through the centre; then, if possible, suppose E to be the middle point of *both*;

that is, let AE = EC; and BE = ED.

Construction. Find F, the centre of the circle. III. 1.

Join EF.

Proof. Because FE, which passes through the centre, bisects the chord AC, *Hyp.*

∴ the ∠ FEC is a rt. angle. III. 3.

And because FE, which passes through the centre, bisects the chord BD, *Hyp.*

∴ the ∠ FED is a rt. angle. III. 3.

∴ the ∠ FEC = the ∠ FED,

the whole equal to its part, which is impossible.

∴ AC and BD are not *both* bisected at E. Q.E.D.

EXERCISES.

ON PROPOSITION 1.

1. If two circles intersect at the points A, B, shew that the line which joins their centres bisects their common chord AB at right angles.

2. AB, AC are two equal chords of a circle; shew that the straight line which bisects the angle BAC passes through the centre.

3. *Two chords of a circle are given in position and magnitude: find the centre of the circle.*

4. *Describe a circle that shall pass through three given points, which are not in the same straight line.*

5. *Find the locus of the centres of circles which pass through two given points.*

6. Describe a circle that shall pass through two given points, and have a given radius. When is this impossible?

ON PROPOSITION 2.

7. *A straight line cannot cut a circle in more than two points.*

ON PROPOSITION 3.

8. Through a given point within a circle draw a chord which shall be bisected at that point.

9. The parts of a straight line intercepted between the circumferences of two concentric circles are equal.

10. The line joining the middle points of two parallel chords of a circle passes through the centre.

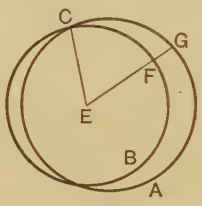
11. Find the locus of the middle points of a system of parallel chords drawn in a circle.

12. If two circles cut one another, any two parallel straight lines drawn through the points of intersection to cut the circles, are equal.

13. PQ and XY are two parallel chords in a circle: shew that the points of intersection of PX, QY, and of PY, QX, lie on the straight line which passes through the middle points of the given chords.

Proposition 5. Theorem.

If two circles cut one another, they cannot have the same centre.



Let the two ⊙ˢ AGC, BFC cut one another at C.

Then they shall not have the same centre.

Construction. If possible, let the two circles have the same centre; and let it be called E.

Join EC;

and from E draw any st. line to meet the ○ces at F and G.

Proof. Because E is the centre of the ⊙ AGC, *Hyp.*

∴ EG = EC.

And because E is also the centre of the ⊙ BFC, *Hyp.*

∴ EF = EC.

∴ EG = EF,

the whole equal to its part, which is impossible.

Therefore the two circles have not the same centre.

Q.E.D.

EXERCISES.

on Propositions 4 and 5.

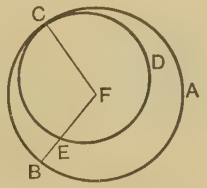
1. If a parallelogram can be inscribed in a circle, the point of intersection of its diagonals must be at the centre of the circle.

2. Rectangles are the only parallelograms that can be inscribed in a circle.

3. Two circles, which intersect at one point, must also intersect at another.

PROPOSITION 6. THEOREM.

If two circles touch one another internally, they cannot have the same centre.



Let the two ⊙ˢ ABC, DEC touch one another internally at C.

Then they shall not have the same centre.

Construction. If possible, let the two circles have the same centre; and let it be called F.

Join FC;

and from F draw any st. line to meet the ○ᶜᵉˢ at E and B.

Proof. Because F is the centre of the ⊙ ABC, *Hyp.*

$\therefore$ FB = FC.

And because F is the centre of the ⊙ DEC, *Hyp.*

$\therefore$ FE = FC.

$\therefore$ FB = FE,

the whole equal to its part, which is impossible.

Therefore the two circles have not the same centre.

Q.E.D.

NOTE. From Propositions 5 and 6 it is seen that circles, whose circumferences have any point in common, cannot be concentric, unless they coincide entirely.

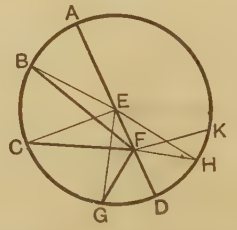
Conversely, the circumferences of concentric circles can have no point in common.

Proposition 7. Theorem.

If from any point within a circle which is not the centre, straight lines are drawn to the circumference, then the greatest is that which passes through the centre; and the least is the remaining part of the diameter.

And of all other such lines, that which is nearer to the greatest is always greater than one more remote.

And two equal straight lines, and only two, can be drawn from the given point to the circumference, one on each side of the diameter.



Let ABCD be a circle, and from F, any point within it which is not the centre, let FA, FB, FC, FG, and FD be drawn to the ○ce, of which FA passes through E the centre, and FD is the remaining part of the diameter.

Then of all these st. lines,

(i) FA *shall be the greatest;*

(ii) FD *shall be the least;*

(iii) FB, *which is nearer to* FA, *shall be greater than* FC, *which is more remote;*

(iv) *also two, and only two, equal st. lines can be drawn from* F *to the* ○ce.

Construction. Join EB, EC.

Proof. (i) In the △ FEB, the two sides FE, EB are together greater than the third side FB. I. 20.

But EB = EA, being radii of the circle;

∴ FE, EA are together greater than FB;

that is, FA is greater than FB.

Similarly FA may be shewn to be greater than any other st. line drawn from F to the $\bigcirc^{ce}$;

$\therefore$ FA is the greatest of all such lines.

(ii) In the $\triangle$ EFG, the two sides EF, FG are together greater than EG; I. 20.

and EG = ED, being radii of the circle;

$\therefore$ EF, FG are together greater than ED.

Take away the common part EF;

then FG is greater than FD.

Similarly any other st. line drawn from F to the $\bigcirc^{ce}$ may be shewn to be greater than FD;

$\therefore$ FD is the least of all such lines.

(iii) In the $\triangle^s$ BEF, CEF,

Because { BE = CE, I. *Def.* 15.
and EF is common;
but the $\angle$ BEF is greater than the $\angle$ CEF;

$\therefore$ FB is greater than FC. I. 24.

Similarly it may be shewn that FC is greater than FG.

(iv) Join EG, and at E in FE make the $\angle$ FEH equal to the $\angle$ FEG. I. 23.

Join FH.

Then in the $\triangle^s$ GEF, HEF,

Because { GE = HE, I. *Def.* 15.
and EF is common;
also the $\angle$ GEF = the $\angle$ HEF; *Constr.*

$\therefore$ FG = FH. I. 4.

And besides FH no other straight line can be drawn from F to the $\bigcirc^{ce}$ equal to FG.

For, if possible, let FK = FG.

Then, because FH = FG, *Proved.*

$\therefore$ FK = FH,

that is, a line nearer to FA, the greatest, is equal to a line which is more remote; which is impossible. *Proved.*

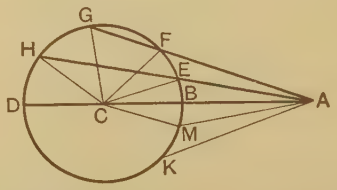
Therefore two, and only two, equal st. lines can be drawn from F to the $\bigcirc^{ce}$. Q.E.D.

PROPOSITION 8. THEOREM.

If from any point without a circle straight lines are drawn to the circumference, of those which fall on the concave circumference, the greatest is that which passes through the centre; and of others, that which is nearer to the greatest is always greater than one more remote.

Of those which fall on the convex circumference, the least is that which, when produced, passes through the centre; and of others, that which is nearer to the least is always less than one more remote.

From the given point there can be drawn to the circumference two, and only two, equal straight lines, one on each side of the shortest line.



Let BGD be a circle; and from A, any point outside the circle, let ABD, AEH, AFG, be drawn, of which AD passes through C, the centre, and AH is nearer than AG to AD.

Then of st. lines drawn from A *to the concave* $\bigcirc^{ce}$,

(i) AD *shall be the greatest, and* (ii) AH *greater than* AG.

And of st. lines drawn from A *to the convex* $\bigcirc^{ce}$,

(iii) AB *shall be the least, and* (iv) AE *less than* AF.

(v) *Also two, and only two, equal st. lines can be drawn from* A *to the* $\bigcirc^{ce}$.

Construction. Join CH, CG, CF, CE.

Proof. (i) In the $\triangle$ ACH, the two sides AC, CH are together greater than AH: I. 20.

but CH = CD, being radii of the circle;

$\therefore$ AC, CD are together greater than AH:

that is, AD is greater than AH.

Similarly AD may be shewn to be greater than any other st. line drawn from A to the concave $\bigcirc^{ce}$;

$\therefore$ AD is the greatest of all such lines.

(ii) In the $\triangle^s$ HCA, GCA,

Because $\begin{cases} \text{HC} = \text{GC}, & \text{I. } \textit{Def.}\ 15. \\ \text{and CA is common;} & \\ \text{but the } \angle \text{ HCA is greater than the } \angle \text{ GCA;} & \end{cases}$

$\therefore$ AH is greater than AG. I. 24.

(iii) In the $\triangle$ AEC, the two sides AE, EC are together greater than AC; I. 20.

but EC = BC; I. *Def.* 15.

$\therefore$ the remainder AE is greater than the remainder AB.

Similarly any other st. line drawn from A to the convex $\bigcirc^{ce}$ may be shewn to be greater than AB;

$\therefore$ AB is the least of all such lines.

(iv) In the $\triangle$ AFC, because AE, EC are drawn from the extremities of the base to a point E within the triangle,

$\therefore$ AF, FC are together greater than AE, EC. I. 21.

But FC = EC; I. *Def.* 15.

$\therefore$ the remainder AF is greater than the remainder AE.

(v) At C, in AC, make the $\angle$ ACM equal to the $\angle$ ACE.

Join AM.

Then in the two $\triangle^s$ ECA, MCA,

Because $\begin{cases} \text{EC} = \text{MC}, & \text{I. } \textit{Def.}\ 15. \\ \text{and CA is common;} & \\ \text{also the } \angle \text{ ECA} = \text{the } \angle \text{ MCA;} & \textit{Constr.} \end{cases}$

$\therefore$ AE = AM. I. 4.

And besides AM, no st. line can be drawn from A to the $\bigcirc^{ce}$, equal to AE.

For, if possible, let AK = AE:

then because AM = AE, *Proved.*

$\therefore$ AM = AK;

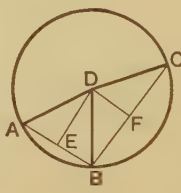
that is, a line nearer to AB, the shortest line, is equal to a line which is more remote; which is impossible. *Proved.*

Therefore two, and only two, equal st. lines can be drawn from A to the $\bigcirc^{ce}$. Q.E.D.

EXERCISE. Where are the limits of that part of the circumference which is concave to the point A?

PROPOSITION 9. THEOREM. [FIRST PROOF.]

If from a point within a circle more than two equal straight lines can be drawn to the circumference, that point is the centre of the circle.



Let ABC be a circle, and D a point within it, from which more than two equal st. lines are drawn to the ○ce, namely DA, DB, DC.

Then D *shall be the centre of the circle* ABC.

Construction. Join AB, BC:
and bisect AB, BC at E and F respectively. I. 10.
Join DE, DF.

Proof. In the △s DEA, DEB,

Because { EA = EB, *Constr.*
and DE is common;
and DA = DB; *Hyp.*

∴ the ∠ DEA = the ∠ DEB; I. 8.

∴ these angles, being adjacent, are rt. angles.

Hence ED, which bisects the chord AB at rt. angles, must pass through the centre. III. 1. *Cor.*

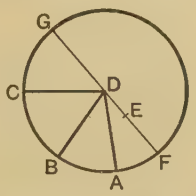
Similarly it may be shewn that FD passes through the centre.

∴ D, which is the only point common to ED and FD, must be the centre. Q.E.D.

NOTE. Of the two proofs of this proposition given by Euclid the first has the advantage of being *direct*.

PROPOSITION 9. THEOREM. [SECOND PROOF.]

If from a point within a circle more than two equal straight lines can be drawn to the circumference, that point is the centre of the circle.



Let ABC be a circle, and D a point within it, from which more than two equal st. lines are drawn to the ○ce, namely DA, DB, DC.

Then D *shall be the centre of the circle* ABC.

Construction. For if not, suppose, if possible, E to be the centre.

Join DE, and produce it to meet the ○ce at F, G.

Proof. Because D is a point within the circle, not the centre, and because DF passes through the centre E;

∴ DA, which is nearer to DF, is greater than DB, which is more remote: III. 7.

but this is impossible, since by hypothesis, DA, DB, are equal.

∴ E is not the centre of the circle.

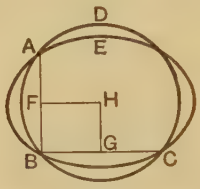
* And wherever we suppose the centre E to be, otherwise than at D, two at least of the st. lines DA, DB, DC may be shewn to be unequal, which is contrary to hypothesis.

∴ D is the centre of the ⊙ ABC. Q.E.D.

* NOTE. For example, if the centre E were supposed to be within the angle BDC, then DB would be greater than DA; if within the angle ADB, then DB would be greater than DC; if on one of the three straight lines, as DB, then DB would be greater than both DA and DC.

PROPOSITION 10. THEOREM. [FIRST PROOF.]

One circle cannot cut another at more than two points.



If possible, let DABC, EABC be two circles, cutting one another at more than two points, namely at A, B, C.

Construction. Join AB, BC.
Draw FH, bisecting AB at rt. angles; I. 10, 11.
and draw GH bisecting BC at rt. angles.

Proof. Because AB is a chord of *both* circles, and because FH bisects AB at rt. angles,
∴ the centre of both circles lies in FH. III. 1. *Cor.*

Again, because BC is a chord of both circles, and because GH bisects BC at right angles,
∴ the centre of both circles lies in GH. III. 1. *Cor.*

Hence H, the only point common to FH and GH, is the centre of both circles;
which is impossible, for circles which cut one another cannot have a common centre. III. 5.

Therefore one circle cannot cut another at more than two points. Q.E.D.

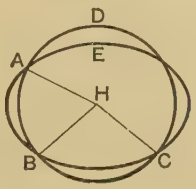
COROLLARIES. (i) *Two circles cannot have three points in common without coinciding entirely.*

(ii) *Two circles cannot have a common arc without coinciding entirely.*

(iii) *Only one circle can be described through three points, which are not in the same straight line.*

PROPOSITION 10. THEOREM. [SECOND PROOF.]

One circle cannot cut another at more than two points.



If possible, let DABC, EABC be two circles, cutting one another at more than two points, namely at A, B, C.

Construction. Find H, the centre of the ⊙ DABC, III. 1.
and join HA, HB, HC.

Proof. Since H is the centre of the ⊙ DABC,
∴ HA, HB, HC are all equal. I. *Def.* 15.

And because H is a point within the ⊙ EABC, from which more than two equal st. lines, namely HA, HB, HC are drawn to the ○ᵉˢ,

∴ H is the centre of the ⊙ EABC: III. 9.

that is to say, the two circles have a common centre H;
but this is impossible, since they cut one another. III. 5.

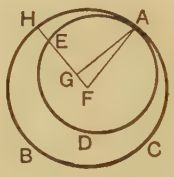
Therefore one circle cannot cut another in more than two points. Q.E.D.

NOTE. Both the proofs of Proposition 10 given by Euclid are indirect.

The second of these is imperfect, because it assumes that the centre of the circle DABC must fall within the circle EABC; whereas it may be conceived to fall either without the circle EABC, or on its circumference. Hence to make the proof complete, two additional cases are required.

Proposition 11. Theorem.

If two circles touch one another internally, the straight line which joins their centres, being produced, shall pass through the point of contact.



Let ABC and ADE be two circles which touch one another internally at A; let F be the centre of the ⊙ ABC, and G the centre of the ⊙ ADE.

Then shall FG *produced pass through* A.

Construction. For if not, suppose, if possible, FG to pass otherwise, as FGEH.

Join FA, GA.

Proof. In the △ FGA, the two sides FG, GA are together greater than FA: I. 20.

but FA = FH, being radii of the ⊙ ABC: *Hyp.*

FG, GA are together greater than FH.

Take away the common part FG:

then GA is greater than GH.

But GA = GE, being radii of the ⊙ ADE: *Hyp.*

∴ GE is greater than GH,

the part greater than the whole; which is impossible.

∴ FG, when produced, must pass through A.

Q.E.D.

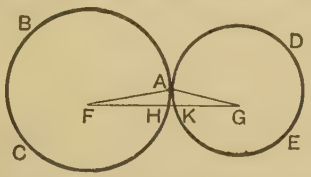
EXERCISES.

1. If the distance between the centres of two circles is equal to the difference of their radii, then the circles must meet in one point, but in no other; that is, they must touch one another.

2. *If two circles whose centres are* A *and* B *touch one another internally, and a straight line is drawn through their point of contact, cutting the circumferences at* P *and* Q; *shew that the radii* AP *and* BQ *are parallel.*

PROPOSITION 12. THEOREM.

If two circles touch one another externally, the straight line which joins their centres shall pass through the point of contact.



Let ABC and ADE be two circles which touch one another externally at A; let F be the centre of the ⊙ ABC, and G the centre of the ⊙ ADE.

Then shall FG *pass through* A.

Construction. For if not, suppose, if possible, FG to pass otherwise, as FHKG.

Join FA, GA.

Proof. In the △ FAG, the two sides FA, GA are together greater than FG: I. 20.

but FA = FH, being radii of the ⊙ ABC; *Hyp.*

and GA = GK, being radii of the ⊙ ADE; *Hyp.*

∴ FH and GK are together greater than FG;

which is impossible.

∴ FG must pass through A.

Q.E.D.

EXERCISES.

1. *Find the locus of the centres of all circles which touch a given circle at a given point.*

2. *Find the locus of the centres of all circles of given radius, which touch a given circle.*

3. If the distance between the centres of two circles is equal to the sum of their radii, then the circles meet in one point, but in no other; that is, they touch one another.

4. *If two circles whose centres are* A *and* B *touch one another externally, and a straight line is drawn through their point of contact cutting the circumferences at* P *and* Q; *shew that the radii* AP *and* BQ *are parallel.*

Proposition 13. Theorem.

Two circles cannot touch one another at more than one point, whether internally or externally.

Fig. 1.

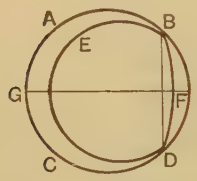
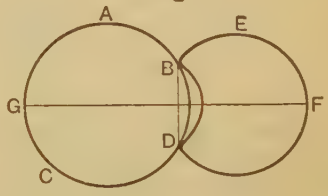


Fig. 2.



If possible, let ABC, EDF be two circles which touch one another at more than one point, namely at B and D.

Construction. Join BD;
and draw GF, bisecting BD at rt. angles. I. 10, 11.

Proof. Now, whether the circles touch one another internally, as in Fig 1 or externally as in Fig 2,
because B and D are on the ○ces of both circles,
∴ BD is a chord of both circles:
∴ the centres of both circles lie in GF, which bisects BD at rt. angles. III. 1. *Cor.*
Hence GF which joins the centres must pass through a point of contact; III. 11, and 12.
which is impossible, since B and D are outside GF.

Therefore two circles cannot touch one another at more than one point.

Q.E.D.

Note. It must be observed that the proof here given applies, by virtue of Propositions 11 and 12, to *both* the above figures: we have therefore omitted the separate discussion of Fig. 2, which finds a place in most editions based on Simson's text.

EXERCISES ON PROPOSITIONS 1–13.

1. Describe a circle to pass through two given points and have its centre on a given straight line. When is this impossible?

2. All circles which pass through a fixed point, and have their centres on a given straight line, pass also through a second fixed point.

3. Describe a circle of given radius to touch a given circle at a given point. How many solutions will there be? When will there be only one solution?

4. From a given point as centre describe a circle to touch a given circle. How many solutions will there be?

5. Describe a circle to pass through a given point, and touch a given circle at a given point. [See Ex. 1, p. 183, and Ex. 5, p. 171.] When is this impossible?

6. Describe a circle of given radius to touch two given circles. [See Ex. 2, p. 183.] How many solutions will there be?

7. Two parallel chords of a circle are six inches and eight inches in length respectively, and the perpendicular distance between them is one inch: find the radius.

8. If two circles touch one another externally, the straight lines, which join the extremities of parallel diameters towards opposite parts, must pass through the point of contact.

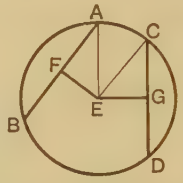
9. Find the greatest and least straight lines which have one extremity on each of two given circles, which do not intersect.

10. In any segment of a circle, of all straight lines drawn at right angles to the chord and intercepted between the chord and the arc, the greatest is that which passes through the middle point of the chord; and of others that which is nearer the greatest is greater than one more remote.

11. If from any point on the circumference of a circle straight lines be drawn to the circumference, the greatest is that which passes through the centre; and of others, that which is nearer to the greatest is greater than one more remote; and from this point there can be drawn to the circumference two, and only two, equal straight lines.

PROPOSITION 14. THEOREM.

Equal chords in a circle are equidistant from the centre.
Conversely, chords which are equidistant from the centre are equal.



Let ABC be a circle, and let AB and CD be chords, of which the perp. distances from the centre are EF and EG.

First. Let AB = CD.

Then shall AB *and* CD *be equidistant from the centre* E.

Construction. Join EA, EC.

Proof. Because EF, which passes through the centre, is perp. to the chord AB; *Hyp.*

∴ EF bisects AB; III. 3.

∴ AB is double of FA.

For a similar reason, CD is double of GC.

But AB = CD, *Hyp.*

∴ FA = GC. *Ax.* 7.

Now EA = EC, being radii of the circle;

∴ the sq. on EA = the sq. on EC.

But since the ∠ EFA is a rt. angle:

∴ the sq. on EA = the sqq. on EF, FA. I. 47.

And since the ∠ EGC is a rt. angle;

∴ the sq. on EC = the sqq. on EG, GC.

∴ the sqq. on EF, FA = the sqq. on EG, GC.

Now of these, the sq. on FA = the sq. on GC; for FA = GC.

∴ the sq. on EF = the sq. on EG;

∴ EF = EG;

that is, the chords AB, CD are equidistant from the centre.

Q.E.D.

Conversely. Let AB and CD be equidistant from the centre E;

that is, let EF = EG.

Then shall AB = CD.

Proof. The same construction being made, it may be shewn as before that AB is double of FA, and CD double of GC;

and that the sqq. on EF, FA = the sqq. on EG, GC.

Now of these, the sq. on EF = the sq. on EG,

for EF = EG: *Hyp.*

∴ the sq. on FA = the sq. on GC;

∴ FA = GC;

and doubles of these equals are equal; *Ax.* 6.

that is, AB = CD.

Q.E.D.

EXERCISES.

1. *Find the locus of the middle points of equal chords of a circle.*

2. If two chords of a circle cut one another, and make equal angles with the straight line which joins their point of intersection to the centre, they are equal.

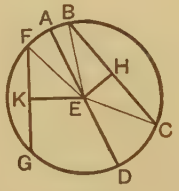
3. *If two equal chords of a circle intersect, shew that the segments of the one are equal respectively to the segments of the other.*

4. In a given circle draw a chord which shall be equal to one given straight line (not greater than the diameter) and parallel to another.

5. PQ is a fixed chord in a circle, and AB is any diameter: shew that the sum or difference of the perpendiculars let fall from A and B on PQ is constant, that is, the same for all positions of AB.

PROPOSITION 15. THEOREM.

The diameter is the greatest chord in a circle;
and of other chords, that which is nearer to the centre is greater than one more remote.
Conversely, the greater chord is nearer to the centre than the less.



Let ABCD be a circle of which AD is a diameter, and E the centre; and let BC and FG be any two chords, whose perp. distances from the centre are EH and EK.

Then (i) AD *shall be greater than* BC;
(ii) *if* EH *is less than* EK, BC *shall be greater than* FG:
(iii) *if* BC *is greater than* FG, EH *shall be less than* EK.

(i) **Construction.** Join EB, EC.

Proof. In the △ BEC, the two sides BE, EC are together greater than BC; I. 20.
but BE = AE, I. *Def.* 15.
and EC = ED;
∴ AE and ED together are greater than BC;
that is, AD is greater than BC.

Similarly AD may be shewn to be greater than any other chord, not a diameter.

(ii) Let EH be less than EK.
Then BC *shall be greater than* FG.

Construction. Join EF.

Proof. Since EH, passing through the centre, is perp. to the chord BC,
∴ EH bisects BC; III. 3.

$\therefore$ BC is double of HB.

For a similar reason FG is double of KF.

Now EB = EF, I. *Def.* 15.

$\therefore$ the sq. on EB = the sq. on EF.

But since the $\angle$ EHB is a rt. angle,

$\therefore$ the sq. on EB = the sqq. on EH, HB. I. 47.

And since the $\angle$ EKF is a rt. angle,

$\therefore$ the sq on EF = the sqq. on EK, KF;

$\therefore$ the sqq. on EH, HB = the sqq. on EK, KF.

But the sq. on EH is less than the sq. on EK,

for EH is less than EK; *Hyp.*

$\therefore$ the sq. on HB is greater than the sq. on KF;

$\therefore$ HB is greater than KF:

hence BC is greater than FG.

(iii) *Conversely.* Let BC be greater than FG.

Then EH *shall be less than* EK.

Proof. The same construction being made, it may be shewn as before that BC is double of BH and FG double of FK; and that the sqq. on EH, HB = the sqq. on EK, KF.

But since BC is greater than FG, *Hyp.*

$\therefore$ HB is greater than KF;

$\therefore$ the sq. on HB is greater than the sq on KF.

$\therefore$ the sq. on EH is less than the sq. on EK;

$\therefore$ EH is less than EK. Q.E.D.

EXERCISES.

1. *Through a given point within a circle draw the least possible chord.*

2. AB is a fixed chord of a circle, and XY any other chord having its middle point Z on AB; what is the greatest, and what the least length that XY may have?

Shew that XY increases, as Z approaches the middle point of AB.

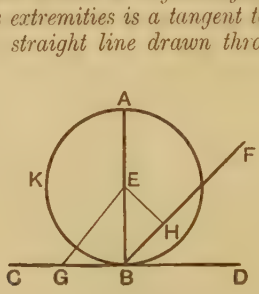
3. In a given circle draw a chord of given length, having its middle point on a given chord.

When is this problem impossible?

PROPOSITION 16. THEOREM. [ALTERNATIVE PROOF.]

The straight line drawn at right angles to a diameter of a circle at one of its extremities is a tangent to the circle:

and every other straight line drawn through this point cuts the circle.



Let AKB be a circle, of which E is the centre, and AB a diameter; and through B let the st. line CBD be drawn at rt. angles to AB.

Then (i) CBD *shall be a tangent to the circle*;

(ii) *any other st. line through* B, *such as* BF, *shall cut the circle.*

(i) Construction. In CD take any point G, and join EG.

Proof. In the $\triangle$ EBG, the $\angle$ EBG is a rt. angle; *Hyp.*

$\therefore$ the $\angle$ EGB is less than a rt. angle; I. 17.

$\therefore$ the $\angle$ EBG is greater than the $\angle$ EGB;

$\therefore$ EG is greater than EB: I. 19.

that is, EG is greater than a radius of the circle;

$\therefore$ the point G is without the circle.

Similarly any other point in CD, except B, may be shewn to be outside the circle.

Hence CD meets the circle at B, but being produced, does not cut it;

that is, CD is a tangent to the circle. III. *Def.* 5.

(ii) Construction. Draw EH perp. to BF. I. 12.

Proof. In the $\triangle$ EHB, because the $\angle$ EHB is a rt. angle,

$\therefore$ the $\angle$ EBH is less than a rt. angle; I. 17.

$\therefore$ EB is greater than EH; I. 19.

that is, EH is less than a radius of the circle:

$\therefore$ H, a point in BF, is within the circle;

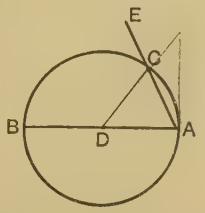
$\therefore$ BF must cut the circle. Q.E.D.

NOTE. The above proof of Proposition 16 is not that given by Euclid, but it is preferable as being *direct*. Euclid's proof by *Reductio ad Absurdum* is given below.

PROPOSITION 16. THEOREM. [EUCLID'S PROOF.]

The straight line drawn at right angles to a diameter of a circle at one of its extremities, is a tangent to the circle:

and no other straight line can be drawn through this point so as not to cut the circle.



Let ABC be a circle, of which D is the centre, and AB a diameter; let AE be drawn at rt. angles to BA, at its extremity A.

(i) *Then shall* AE *be a tangent to the circle.*

Construction.

For, if possible, suppose AE to cut the circle at C.

Join DC.

Proof. Then in the $\triangle$ DAC, because DA = DC, I. *Def.* 15.

$\therefore$ the $\angle$ DAC = the $\angle$ DCA.

But the $\angle$ DAC is a rt. angle; *Hyp.*

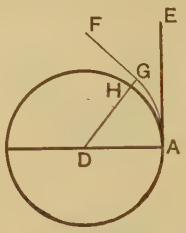
$\therefore$ the $\angle$ DCA is a rt. angle;

that is, two angles of the $\triangle$ DAC are together equal to two rt. angles; which is impossible. I. 17.

Hence AE meets the circle at A, but being produced, does not cut it;

that is, AE is a tangent to the circle. III. *Def.* 5.

(ii) *Also through* A *no other straight line but* AE *can be drawn so as not to cut the circle.*



Construction. For, if possible, let AF be another st. line drawn through A so as not to cut the circle.

From D draw DG perp. to AF ; 1. 12.
and let DG meet the ○ce at H.

Proof. Then in the △ DAG, because the ∠ DGA is a rt. angle,
∴ the ∠ DAG is less than a rt. angle ; 1. 17.
∴ DA is greater than DG. 1. 19.
But DA = DH, 1. *Def.* 15.
∴ DH is greater than DG,
the part greater than the whole, which is impossible.

Therefore no st. line can be drawn from the point A, so as not to cut the circle, except AE.

COROLLARY. (i) *A tangent touches a circle at one point only.*

COROLLARY. (ii) *There can be but one tangent to a circle at a given point.*

PROPOSITION 17. PROBLEM.

To draw a tangent to a circle from a given point either on, or without the circumference.

Fig. 1.

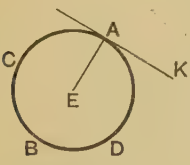
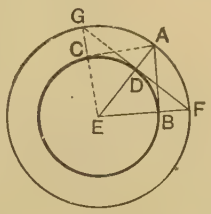


Fig. 2.



Let BCD be the given circle, and A the given point.

It is required to draw from A *a tangent to the* ⊙ CDB.

CASE I. When the given point A is on the ○ce.

Construction. Find E, the centre of the circle. III. 1.

Join EA.

At A draw AK at rt. angles to EA. I. 11.

Proof. Then AK being perp to a diameter at one of its extremities, is a tangent to the circle. III. 16.

CASE II. When the given point A is without the ○ce.

Construction. Find E, the centre of the circle; III. 1.

and join AE, cutting the ⊙ BCD at D.

With centre E and radius EA, describe the ⊙ AFG.

At D, draw GDF at rt. angles to EA, cutting the ⊙ AFG at F and G. I. 11.

Join EF, EG, cutting the ⊙ BCD at B and C.

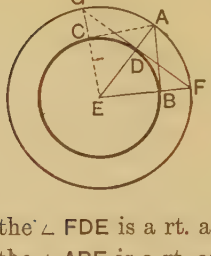
Join AB, AC.

Then both AB *and* AC *shall be tangents to the* ⊙ CDB.

Proof. In the △s AEB, FED,

Because { AE = FE, being radii of the ⊙ GAF;
and EB = ED, being radii of the ⊙ BDC;
and the included angle AEF is common;

∴ the ∠ ABE = the ∠ FDE. I. 4.



But the ∠ FDE is a rt. angle; *Constr.*

∴ the ∠ ABE is a rt. angle.

Hence AB, being drawn at rt. angles to a diameter at one of its extremities, is a tangent to the ⊙ BCD. III. 16.

Similarly it may be shewn that AC is a tangent. Q.E.F.

COROLLARY. *If two tangents are drawn to a circle from an external point, then* (i) *they are equal;* (ii) *they subtend equal angles at the centre;* (iii) *they make equal angles with the straight line which joins the given point to the centre.*

For, in the above figure,

Since ED is perp. to FG, a chord of the ⊙ FAG, *Constr.*

∴ DF = DG. III. 3.

Then in the △ˢ DEF, DEG,

Because { DE is common to both,
and EF = EG; I. *Def.* 15.
and DF = DG; *Proved.*

∴ the ∠ DEF = the ∠ DEG. I. 8.

Again in the △ˢ AEB, AEC,

Because { AE is common to both,
and EB = EC,
and the ∠ AEB = the ∠ AEC; *Proved.*

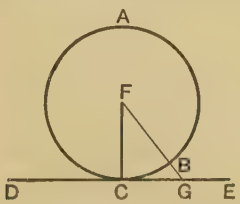
∴ AB = AC: I. 4.

and the ∠ EAB = the ∠ EAC. Q.E.D.

NOTE. If the given point A is within the circle, no solution is possible. Hence we see that this problem admits of *two* solutions, *one* solution, or *no* solution, according as the given point A is *without*, *on*, or *within* the circumference of a circle. For a simpler method of drawing a tangent to a circle from a given point, see page 218.

PROPOSITION 18. THEOREM.

The straight line drawn from the centre of a circle to the point of contact of a tangent is perpendicular to the tangent.



Let ABC be a circle, of which F is the centre;
and let the st. line DE touch the circle at C.

Then shall FC *be perp. to* DE.

For, if not, suppose, if possible, FG to be perp. to DE, I. 12.
and let FG meet the $\bigcirc^{ce}$ at B.

Proof.

In the $\triangle$ FCG, because the $\angle$ FGC is a rt. angle, *Hyp.*
$\therefore$ the $\angle$ FCG is less than a rt. angle; I. 17.
$\therefore$ the $\angle$ FGC is greater than the $\angle$ FCG;
$\therefore$ FC is greater than FG: I. 19.
but FC = FB;
$\therefore$ FB is greater than FG,
the part greater than the whole, which is impossible.
$\therefore$ FC cannot be otherwise than perp. to DE:
that is, FC is perp. to DE. Q.E.D.

EXERCISES.

1. Draw a tangent to a circle (i) parallel to, (ii) at right angles to a given straight line.

2. *Tangents drawn to a circle from the extremities of a diameter are parallel.*

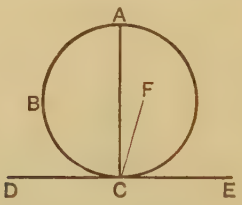
3. *Circles which touch one another internally or externally have a common tangent at their point of contact.*

4. *In two concentric circles, any chord of the outer circle which touches the inner, is bisected at the point of contact.*

5. *In two concentric circles, all chords of the outer circle which touch the inner, are equal.*

PROPOSITION 19. THEOREM.

The straight line drawn perpendicular to a tangent to a circle from the point of contact passes through the centre.



Let ABC be a circle, and DE a tangent to it at the point C; and let CA be drawn perp. to DE.

Then shall CA *pass through the centre.*

Construction. For if not, suppose, if possible, the centre F to be outside CA.

Join CF.

Proof. Because DE is a tangent to the circle, and FC is drawn from the centre F to the point of contact,

$\therefore$ the $\angle$ FCE is a rt. angle. III. 18.

But the $\angle$ ACE is a rt. angle; *Hyp.*

$\therefore$ the $\angle$ FCE = the $\angle$ ACE;

the part equal to the whole, which is impossible.

$\therefore$ the centre cannot be otherwise than in CA;

that is, CA passes through the centre.

Q.E.D.

EXERCISES ON THE TANGENT.

PROPOSITIONS 16, 17, 18, 19.

1. *The centre of any circle which touches two intersecting straight lines must lie on the bisector of the angle between them.*

2. AB and AC are two tangents to a circle whose centre is O; shew that AO bisects the chord of contact BC at right angles.

3. If two circles are concentric all tangents drawn from points on the circumference of the outer to the inner circle are equal.

4. The diameter of a circle bisects all chords which are parallel to the tangent at either extremity.

5. *Find the locus of the centres of all circles which touch a given straight line at a given point.*

6. Find the locus of the centres of all circles which touch each of two parallel straight lines.

7. *Find the locus of the centres of all circles which touch each of two intersecting straight lines of unlimited length.*

8. Describe a circle of given radius to touch two given straight lines.

9. Through a given point, within or without a circle, draw a chord equal to a given straight line.

In order that the problem may be possible, between what limits must the given line lie, when the given point is (i) without the circle, (ii) within it?

10. Two parallel tangents to a circle intercept on any third tangent a segment which subtends a right angle at the centre.

11. *In any quadrilateral circumscribed about a circle, the sum of one pair of opposite sides is equal to the sum of the other pair.*

12. Any parallelogram which can be circumscribed about a circle, must be equilateral.

13. If a quadrilateral is described about a circle, the angles subtended at the centre by any two opposite sides are together equal to two right angles.

14. AB is any chord of a circle, AC the diameter through A, and AD the perpendicular on the tangent at B: shew that AB bisects the angle DAC.

15. Find the locus of the extremities of tangents of fixed length drawn to a given circle.

16. In the diameter of a circle produced, determine a point such that the tangent drawn from it shall be of given length.

17. In the diameter of a circle produced, determine a point such that the two tangents drawn from it may contain a given angle.

18. Describe a circle that shall pass through a given point, and touch a given straight line at a given point. [See page 197. Ex. 5.]

19. Describe a circle of given radius, having its centre on a given straight line, and touching another given straight line.

20. Describe a circle that shall have a given radius, and touch a given circle and a given straight line. How many such circles can be drawn?

PROPOSITION 20. THEOREM.

The angle at the centre of a circle is double of an angle at the circumference, standing on the same arc.

Fig. 1.

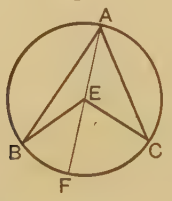
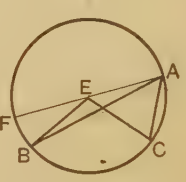


Fig. 2.



Let ABC be a circle, of which E is the centre; and let BEC be the angle at the centre, and BAC an angle at the O^{ce}, standing on the same arc BC.

Then shall the $\angle$ BEC *be double of the* $\angle$ BAC.

Construction. Join AE, and produce it to F.

CASE I. When the centre E is within the angle BAC.

Proof. In the $\triangle$ EAB, because EA = EB, I. *Def.* 15.

$\therefore$ the $\angle$ EAB = the $\angle$ EBA; I. 5.

$\therefore$ the sum of the $\angle^s$ EAB, EBA = twice the $\angle$ EAB.

But the ext. $\angle$ BEF = the sum of the $\angle^s$ EAB, EBA; I. 32.

$\therefore$ the $\angle$ BEF = twice the $\angle$ EAB.

Similarly the $\angle$ FEC = twice the $\angle$ EAC.

$\therefore$ the sum of the $\angle^s$ BEF, FEC = twice the sum of the $\angle^s$ EAB, EAC;

that is, the $\angle$ BEC = twice the $\angle$ BAC.

CASE II. When the centre E is without the $\angle$ BAC.

As before, it may be shewn that the $\angle$ FEB = twice the $\angle$ FAB;

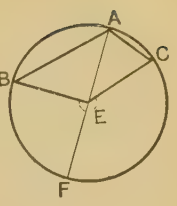
also the $\angle$ FEC = twice the $\angle$ FAC;

$\therefore$ the difference of the $\angle^s$ FEC, FEB = twice the difference of the $\angle^s$ FAC, FAB:

that is, the $\angle$ BEC = twice the $\angle$ BAC. Q.E.D.

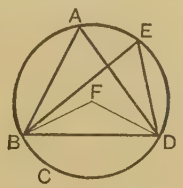
NOTE 1. The case in which the centre E falls *on* AB or AC needs no proof beyond that given under Case I.

NOTE 2. If the arc BFC, on which the angles stand, is greater than a semi-circumference, the angle BEC at the centre will be *reflex*: but it may still be shewn as, in Case I., that the reflex ∠ BEC is double of the ∠ BAC at the ○ce, standing on the same arc BFC.



PROPOSITION 21. THEOREM.

Angles in the same segment of a circle are equal.



Let ABCD be a circle, and let BAD, BED be angles in the same segment BAED.

Then shall the ∠ BAD = *the* ∠ BED.

CONSTRUCTION. Find F, the centre of the circle. III. 1.

CASE I. When the segment BAED is greater than a semicircle.

Join BF, DF.

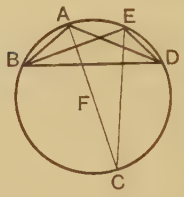
PROOF. Because the ∠ BFD is at the centre, and the ∠ BAD at the ○ce, standing on the same arc BD,

∴ the ∠ BFD = twice the ∠ BAD. III. 20.

Similarly the ∠ BFD = twice the ∠ BED. III. 20.

∴ the ∠ BAD = the ∠ BED. *Ax.* 7.

CASE II. When the segment BAED is not greater than a semicircle.



Construction.

Join AF, and produce it to meet the $\bigcirc^{ce}$ at C.

Join EC.

Proof. Then since AEDC is a semicircle;

$\therefore$ the segment BAEC is greater than a semicircle:

$\therefore$ the $\angle$ BAC = the $\angle$ BEC, in this segment. *Case* 1

Similarly the segment CAED is greater than a semicircle;

$\therefore$ the $\angle$ CAD = the $\angle$ CED, in this segment.

$\therefore$ the $\angle^s$ BAC, CAD = the sum of the $\angle^s$ BEC, CED.

that is, the $\angle$ BAD = the $\angle$ BED. Q.E.D.

EXERCISES.

1. P is any point on the arc of a segment of which AB is the chord. Shew that the sum of the angles PAB, PBA is constant.

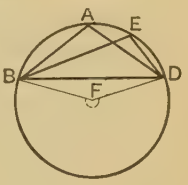
2. PQ and RS are two chords of a circle intersecting at X: prove that the triangles PXS, RXQ are equiangular.

3. Two circles intersect at A and B; and through A any straight line PAQ is drawn terminated by the circumferences: shew that PQ subtends a constant angle at B.

4. Two circles intersect at A and B; and through A any two straight lines PAQ, XAY are drawn terminated by the circumferences; shew that the arcs PX, QY subtend equal angles at B.

5. P is any point on the arc of a segment whose chord is AB: and the angles PAB, PBA are bisected by straight lines which intersect at O. Find the locus of the point O.

NOTE. If the extension of Proposition 20, given in Note 2 on page 199, is adopted, a separate treatment of the second case of the present proposition is unnecessary.



For, as in Case I.,

the reflex $\angle BFD$ = twice the $\angle BAD$;

III. 20.

also the reflex $\angle BFD$ = twice the $\angle BED$;

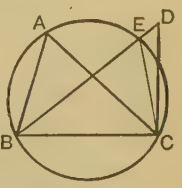
$\therefore$ the $\angle BAD$ = the $\angle BED$.

Obs. The converse of Prop. 21 is important. For the construction used, viz. *To describe a circle about a given triangle*, see Book IV., Prop. 5, or Theorems and Examples on Book I, page 111, No. 1.

CONVERSE OF PROPOSITION 21.

Equal angles standing on the same base, and on the same side of it, have their vertices on an arc of a circle, of which the given base is the chord.

Let BAC, BDC be two equal angles standing on the same base BC.



Then shall the vertices A *and* D *lie upon a segment of a circle having* BC *as its chord.*

Describe a circle about the $\triangle BAC$. IV. 5.

Then this circle shall pass through D.

For, if not, it must cut BD, or BD produced, at some other point E.

Join EC.

Then the $\angle BAC$ = the $\angle BEC$, in the same segment : III. 21.

but the $\angle BAC$ = the $\angle BDC$, by hypothesis ;

$\therefore$ the $\angle BEC$ = the $\angle BDC$;

that is, an ext. angle of a triangle = an int. opp. angle ;

which is impossible. I. 16.

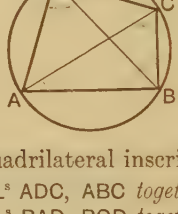
$\therefore$ the circle which passes through B, A, C, cannot pass otherwise than through D.

That is, the vertices A and D are on an arc of a circle of which the chord is BC. Q.E.D.

COROLLARY. *The locus of the vertices of triangles drawn on the same base and on the same side of it with equal vertical angles is an arc of a circle.*

PROPOSITION 22. THEOREM.

The opposite angles of any quadrilateral inscribed in a circle are together equal to two right angles.



Let ABCD be a quadrilateral inscribed in the ⊙ ABC.

Then shall (i) *the* ∠ˢ ADC, ABC *together* = *two rt. angles*;
(ii) *the* ∠ˢ BAD, BCD *together* = *two rt. angles.*

Construction. Join AC, BD.

Proof.

Since the ∠ ADB = the ∠ ACB, in the segment ADCB; III. 21.
and the ∠ CDB = the ∠ CAB, in the segment CDAB;
∴ the ∠ ADC = the sum of the ∠ˢ ACB, CAB.

To each of these equals add the ∠ ABC:
then the two ∠ˢ ADC, ABC together = the three ∠ˢ ACB, CAB, ABC.

But the ∠ˢ ACB, CAB, ABC, being the angles of a triangle, together = two rt. angles; I. 32.
∴ the ∠ˢ ADC, ABC together = two rt. angles.
Similarly it may be shewn that
the ∠ˢ BAD, BCD together = two rt. angles. Q.E.D.

EXERCISES.

1. If a circle can be described about a parallelogram, the parallelogram must be rectangular.

2. ABC is an isosceles triangle, and XY is drawn parallel to the base BC cutting the sides in X and Y: shew that the four points B, C, X, Y lie on a circle.

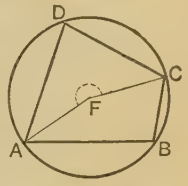
3. *If one side of a quadrilateral inscribed in a circle is produced, the exterior angle is equal to the opposite interior angle of the quadrilateral.*

PROPOSITION 22. [ALTERNATIVE PROOF.]

Let ABCD be a quadrilateral inscribed in the ⊙ ABC.

Then shall the ∠ˢ ADC, ABC *together = two rt. angles.*

Join FA, FC.



Since the ∠ AFC at the centre = twice the ∠ ADC at the ○ce, standing on the same arc ABC; III. 20.

and the reflex angle AFC at the centre = twice the ∠ ABC at the ○ce, standing on the same arc ADC; III. 20.

∴ the ∠ˢ ADC, ABC are together half the sum of the ∠ AFC and the reflex angle AFC;

but these make up four rt. angles: I. 15. *Cor.* 2.

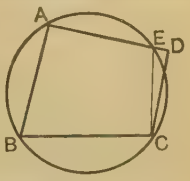
∴ the ∠ˢ ADC, ABC together = two rt. angles. Q.E.D.

DEFINITION. Four or more points through which a circle may be described are said to be **concyclic.**

CONVERSE OF PROPOSITION 22.

If a pair of opposite angles of a quadrilateral are together equal to two right angles, its vertices are concyclic.

Let ABCD be a quadrilateral, in which the opposite angles at B and D together = two rt. angles.



Then shall the four points A, B, C, D *be concyclic.*

Through the three points A, B, C describe a circle. IV. 5.

Then this circle must pass through D.

For, if not, it will cut AD, or AD produced, at some other point E.

Join EC.

Then since the quadrilateral ABCE is inscribed in a circle,

∴ the ∠ˢ ABC, AEC together = two rt. angles. III. 22.

But the ∠ˢ ABC, ADC together = two rt. angles; *Hyp.*

hence the ∠ˢ ABC, AEC = the ∠ˢ ABC, ADC.

Take from these equals the ∠ ABC;

then the ∠ AEC = the ∠ ADC;

that is, an ext. angle of a triangle = an int. opp. angle; which is impossible. I. 16.

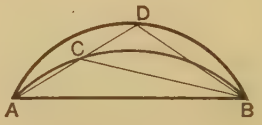
∴ the circle which passes through A, B, C cannot pass otherwise than through D:

that is the four vertices A, B, C, D are concyclic. Q.E.D.

DEFINITION. **Similar segments** of circles are those which contain equal angles. [Book III., Def. 10.]

PROPOSITION 23. THEOREM.

On the same chord and on the same side of it, there cannot be two similar segments of circles, not coinciding with one another.



If possible, on the same chord AB, and on the same side of it, let there be two similar segments of circles ACB, ADB, not coinciding with one another.

Then since the arcs ADB, ACB intersect at A and B,

∴ they cannot cut one another again ; III. 10.

∴ one segment falls within the other.

Construction. In the inner arc take any point C.
Join AC, producing it to meet the outer arc at D :
join CB, DB.

Proof. Then because the segments are similar,

∴ the ∠ ACB = the ∠ ADB ; III. *Def.* 10.

that is, an ext. angle of the △ CDB = an int. opp. angle ; I. 16.
which is impossible.

Hence the two similar segments ACB, ADB, on the same chord AB and on the same side of it, must coincide. Q.E.D.

EXERCISES ON PROPOSITION 22.

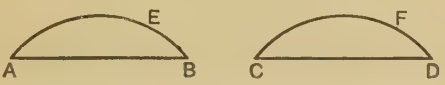
1. The straight lines which bisect any angle of a quadrilateral figure inscribed in a circle and the opposite exterior angle, meet on the circumference.

2. A triangle is inscribed in a circle : shew that the sum of the angles in the three segments exterior to the triangle is equal to four right angles.

3. Divide a circle into two segments, so that the angle contained by the one shall be double of the angle contained by the other.

PROPOSITION 24. THEOREM.

Similar segments of circles on equal chords are equal to one another.



Let AEB and CFD be similar segments on equal chords AB, CD.

Then shall the segment AEB = *the segment* CFD.

Proof. If the segment AEB be applied to the segment CFD, so that A falls on C, and AB falls along CD;

then since AB = CD, *Hyp.*

∴ B must coincide with D.

∴ the segment AEB must coincide with the segment CFD; for if not, on the same chord and on the same side of it there would be two similar segments of circles, not coinciding with one another: which is impossible. III. 23.

∴ the segment AEB = the segment CFD. Q.E.D.

EXERCISES.

1. Of two segments standing on the same chord, the greater segment contains the smaller angle.

2. A segment of a circle stands on a chord AB, and P is any point on the same side of AB as the segment: shew that the angle APB is greater or less than the angle in the segment, according as P is within or without the segment.

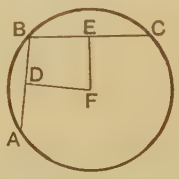
3. P, Q, R *are the middle points of the sides of a triangle, and* X *is the foot of the perpendicular let fall from one vertex on the opposite side: shew that the four points* P, Q, R, X *are concyclic.*

[See page 104, Ex. 2: also page 108, Ex. 2.]

4. *Use the preceding exercise to shew that the middle points of the sides of a triangle and the feet of the perpendiculars let fall from the vertices on the opposite sides, are concyclic.*

PROPOSITION 25. PROBLEM.

An arc of a circle being given, to describe the whole circumference of which the given arc is a part.



Let ABC be an arc of a circle.

It is required to describe the whole ○ce *of which the arc* ABC *is a part.*

Construction.

In the given arc take any three points A, B, C.
Join AB, BC.
Draw DF bisecting AB at rt. angles, I. 10. 11.
and draw EF bisecting BC at rt. angles.

Proof.

Then because DF bisects the chord AB at rt. angles,
∴ the centre of the circle lies in DF. III. 1 *Cor.*
Again, because EF bisects the chord BC at rt. angles,
∴ the centre of the circle lies in EF. III. 1 *Cor.*

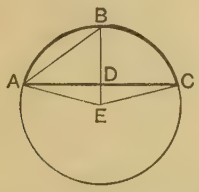
∴ the centre of the circle is F, the only point common to DF and EF.

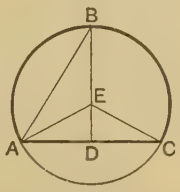
Hence the ○ce of a circle described with centre F, and radius FA, is that of which the given arc is a part. Q.E.F.

NOTE. Euclid gave to this proposition a somewhat different form, as follows:

PROPOSITION 25. [EUCLID'S METHOD.]

A segment of a circle being given, to describe the circle of which it is a segment.





Let ABC be the given segment of a circle, standing on the chord AC.

It is required to describe the circle of which ABC *is a segment.*

Construction. Draw DB, bisecting AC at rt. angles, and meeting the ○ce at B.

Join AB.

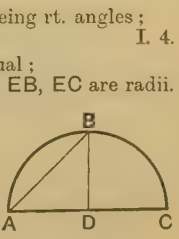
CASE I. When the ∠ DAB is not equal to the ∠ ABD.
At A, in BA, make the ∠ BAE equal to the ∠ ABD; I. 23.
and let AE meet BD, or BD produced, at E.
Join EC.

Then E *shall be the centre of the required circle.*

Proof. Since the ∠ EAB = the ∠ EBA, *Constr.*
∴ EA = EB. I. 6.
And in the △s EDA, EDC,
Because { DA = DC, *Constr.*
and ED is common;
also the ∠ EDA = the ∠ EDC, being rt. angles;
∴ EA = EC. I. 4.

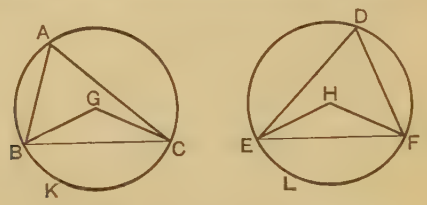
Hence EA, EB, and EC are all equal;
∴ E is the centre of the required circle, and EA, EB, EC are radii.

CASE II. When the ∠ DAB = the ∠ ABD.
In this case it follows that DB = DA; I. 6.
∴ DB, DA, DC are all equal, so that D is the centre of the required circle.



Proposition 26. Theorem.

In equal circles the arcs which subtend equal angles, whether at the centres or at the circumferences, shall be equal.



Let ABC, DEF be equal circles;
and let the $\angle^s$ BGC, EHF at the centres be equal,
and consequently the $\angle^s$ BAC, EDF at the $\bigcirc^{ces}$ equal. III. 20.
Then shall the arc BKC = *the arc* ELF.

Construction. Join BC, EF.

Proof. Because the $\odot^s$ ABC, DEF are equal,
$\therefore$ their radii are equal.
Hence in the $\triangle^s$ BGC, EHF,

Because { BG = EH,
and GC = HF,
and the $\angle$ BGC = the $\angle$ EHF; *Hyp.*
$\therefore$ BC = EF. I. 4.

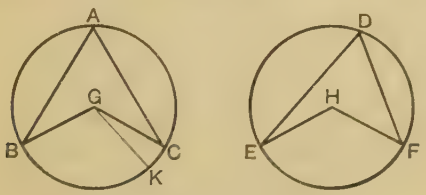
Again, because the $\angle$ BAC = the $\angle$ EDF, *Hyp.*
$\therefore$ the segment BAC is similar to the segment EDF;
III. *Def.* 10.
and these segments are on equal chords BC, EF;
$\therefore$ the segment BAC = the segment EDF. III. 24.
But the whole $\odot$ ABC = the whole $\odot$ DEF;
$\therefore$ the remaining segment BKC = the remaining segment ELF;
$\therefore$ the arc BKC = the arc ELF.

Q.E.D.

[For Exercises and an Alternative Proof see pp. 212, 213.]

PROPOSITION 27. THEOREM.

In equal circles the angles, whether at the centres or the circumferences, which stand on equal arcs, shall be equal.



Let ABC, DEF be equal circles;
and let the arc BC = the arc EF.

Then shall the ∠ BGC = *the* ∠ EHF, *at the centres;*
and also the ∠ BAC = *the* ∠ EDF, *at the* ○ces.

Construction. If the ∠s BGC, EHF are not equal, one must be the greater.

If possible, let the ∠ BGC be the greater.
At G, in BG, make the ∠ BGK equal to the ∠ EHF. I. 23.

Proof. In the equal ⊙s ABC, DEF,
because the ∠ BGK = the ∠ EHF, at the centres; *Constr.*
∴ the arc BK = the arc EF. III. 26.

But the arc BC = the arc EF; *Hyp.*
∴ the arc BK = the arc BC,
a part equal to the whole, which is impossible.

∴ the ∠ BGC is not unequal to the ∠ EHF;
that is, the ∠ BGC = the ∠ EHF.

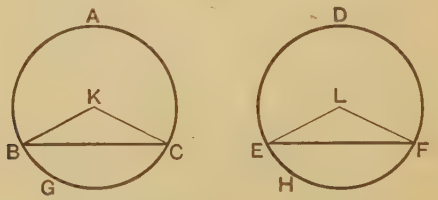
And since the ∠ BAC at the ○ce is half the ∠ BGC at the centre, III. 20.

and likewise the ∠ EDF is half the ∠ EHF,
∴ the ∠ BAC = the ∠ EDF. *Ax.* 7.

Q.E.D.

Proposition 28. Theorem.

In equal circles the arcs, which are cut off by equal chords, shall be equal, the major arc equal to the major arc, and the minor to the minor.



Let ABC, DEF be equal circles ;
and let the chord BC = the chord EF.

Then shall the major arc BAC = *the major arc* EDF ;
and the minor arc BGC = *the minor arc* EHF.

Construction.

Find K and L the centres of the ⊙ˢ ABC, DEF ; III. 1.
and join BK, KC, EL, LF.

Proof. Because the ⊙ˢ ABC, DEF are equal,
∴ their radii are equal.

Hence in the △ˢ BKC, ELF,

Because { BK = EL,
KC = LF,
and BC = EF ; *Hyp.*

∴ the ∠ BKC = the ∠ ELF. I. 8.

∴ the arc BGC = the arc EHF ;

for these arcs subtend equal angles at the centre ; III. 26.
and they are the minor arcs.

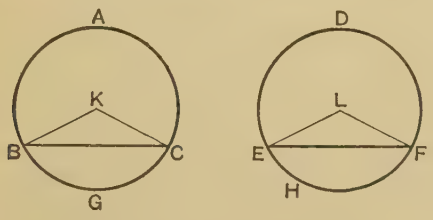
But the whole ○ᶜᵉ ABGC = the whole ○ᶜᵉ DEHF ; *Hyp.*

∴ the remaining arc BAC = the remaining arc EDF :
and these are the major arcs. Q.E.D.

[For Exercises see p. 212.]

PROPOSITION 29. THEOREM.

In equal circles the chords, which cut off equal arcs, shall be equal.



Let ABC, DEF be equal circles;
and let the arc BGC = the arc EHF.
Then shall the chord BC = *the chord* EF.

Construction. Find K, L the centres of the circles.
Join BK, KC, EL, LF.

Proof. In the equal ⊙ˢ ABC, DEF,
because the arc BGC = the arc EHF,
∴ the ∠ BKC = the ∠ ELF, at the centres. III. 27.

Hence in the △ˢ BKC, ELF,
Because { BK = EL, being radii of equal circles;
KC = LF, for the same reason,
and the ∠ BKC = the ∠ ELF; *Proved.*
∴ BC = EF. I. 4.
Q.E.D.

EXERCISES.

ON PROPOSITIONS 26, 27.

1. *If two chords of a circle are parallel, they intercept equal arcs.*

2. The straight lines, which join the extremities of two equal arcs of a circle towards the same parts, are parallel.

3. *In a circle, or in equal circles, sectors are equal if their angles at the centres are equal.*

4. If two chords of a circle intersect at right angles, the opposite arcs are together equal to a semi-circumference.

5. *If two chords intersect within a circle, they form an angle equal to that subtended at the circumference by the sum of the arcs they cut off.*

6. *If two chords intersect without a circle, they form an angle equal to that subtended at the circumference by the difference of the arcs they cut off.*

7. *If* AB *is a fixed chord of a circle, and* P *any point on one of the arcs cut off by it, then the bisector of the angle* APB *cuts the conjugate arc in the same point, whatever be the position of* P.

8. Two circles intersect at A and B; and through these points straight lines are drawn from any point P on the circumference of one of the circles: shew that when produced they intercept on the other circumference an arc which is constant for all positions of P.

9. A triangle ABC is inscribed in a circle, and the bisectors of the angles meet the circumference at X, Y, Z. Find each angle of the triangle XYZ in terms of those of the original triangle.

ON PROPOSITIONS 28, 29.

10. The straight lines which join the extremities of parallel chords in a circle (i) towards the same parts, (ii) towards opposite parts, are equal.

11. Through A, a point of intersection of two equal circles, two straight lines PAQ, XAY are drawn: shew that the chord PX is equal to the chord QY.

12. Through the points of intersection of two circles two parallel straight lines are drawn terminated by the circumferences: shew that the straight lines which join their extremities towards the same parts are equal.

13. Two equal circles intersect at A and B; and through A any straight line PAQ is drawn terminated by the circumferences: shew that BP = BQ.

14. ABC is an isosceles triangle inscribed in a circle, and the bisectors of the base angles meet the circumference at X and Y. Shew that the figure BXAYC must have four of its sides equal.

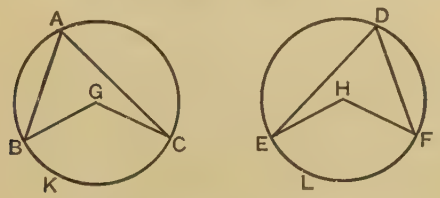
What relation must subsist among the angles of the triangle ABC, in order that the figure BXAYC may be equilateral?

NOTE. We have given Euclid's demonstrations of Propositions 26, 27; but it should be noticed that these propositions also admit of proof by the method of *superposition*.

To illustrate this method we will apply it to Proposition 26.

PROPOSITION 26. [ALTERNATIVE PROOF.]

In equal circles, the arcs which subtend equal angles, whether at the centres or circumferences, shall be equal.



Let ABC, DEF be equal circles;
and let the $\angle^s$ BGC, EHF at the centres be equal,
and consequently the $\angle^s$ BAC, EDF at the $\bigcirc^{ces}$ equal. III. 20.
Then shall the arc BKC = *the arc* ELF.

Proof. For if the $\odot$ ABC be applied to the $\odot$ DEF, so that the centre G may fall on the centre H,

then because the circles are equal, *Hyp.*
$\therefore$ their $\bigcirc^{ces}$ must coincide;

hence by revolving the upper circle about its centre, the lower circle remaining fixed,

B may be made to coincide with E,
and consequently GB with HE.

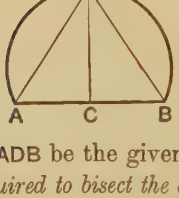
And because the $\angle$ BGC = the $\angle$ EHF, *Hyp.*
$\therefore$ GC must coincide with HF:
and since GC = HF, *Hyp.*
$\therefore$ C must fall on F.

Now B coincides with E, and C with F,
and the $\bigcirc^{ce}$ of the $\odot$ ABC with the $\bigcirc^{ce}$ of the $\odot$ DEF;
$\therefore$ the arc BKC must coincide with the arc ELF.
$\therefore$ the arc BKC = the arc ELF.

Q. E. D.

PROPOSITION 30. PROBLEM.

To bisect a given arc.



Let ADB be the given arc.

It is required to bisect the arc ADB.

Construction. Join AB; and bisect AB at C. I. 10.

At C draw CD at rt. angles to AB, meeting the given arc at D. I. 11.

Then shall the arc ADB *be bisected at* D.

Join AD, BD.

Proof. In the $\triangle^s$ ACD, BCD,

Because { AC = BC, *Constr.*
and CD is common;
and the $\angle$ ACD = the $\angle$ BCD, being rt. angles:

$\therefore$ AD = BD. I. 4.

And since in the $\odot$ ADB, the chords AD, BD are equal,

$\therefore$ the arcs cut off by them are equal, the minor arc equal to the minor, and the major arc to the major: III. 28.

and the arcs AD, BD are both minor arcs,

for each is less than a semi-circumference, since DC, bisecting the chord AB at rt. angles, must pass through the centre of the circle. III. 1. *Cor.*

$\therefore$ the arc AD = the arc BD:

that is, the arc ADB is bisected at D. Q.E.F.

EXERCISES.

1. If a tangent to a circle is parallel to a chord, the point of contact will bisect the arc cut off by the chord.

2. Trisect a quadrant, or the fourth part of the circumference, of a circle.

NOTE. The following alternative proof of Proposition 30 removes the necessity of distinguishing between the major and minor arcs cut off by the chords AD, BD.

PROPOSITION 30. [ALTERNATIVE PROOF.]

The construction being made as before, we may proceed thus:

Proof. In the $\triangle^s$ ACD, BCD,

Because { AC = BC, *Constr.*
and CD is common;
and the $\angle$ ACD = the $\angle$ BCD, being rt. angles:

$\therefore$ the $\angle$ DAC = the $\angle$ DBC: I. 4.

that is, the $\angle$ DAB = the $\angle$ DBA.

But these are angles at the $\bigcirc^{ce}$ subtended by the arcs DB, DA;

$\therefore$ the arc DB = the arc DA: III. 26.

that is, the arc ADB is bisected at D. Q.E.F.

QUESTIONS FOR REVISION.

1. When is a straight line said (i) *to meet*, (ii) *to cut*, (iii) *to touch*, the circumference of a circle?

2. When are circles said *to touch* one another? Distinguish between *internal* and *external* contact.

3. What theorems have been so far proved by Euclid regarding (i) *circles which cut one another*, (ii) *circles which touch one another*?

4. If two unequal circles are concentric, shew that one must lie wholly within the other.

5. Shew how to divide the circumference of a circle into *three*, *four*, or *six* equal parts.

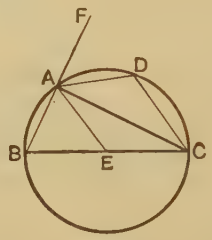
6. Enunciate the propositions so far proved by Euclid relating to the properties of *a tangent to a circle*.

Proposition 31. Theorem.

The angle in a semicircle is a right angle.

The angle in a segment greater than a semicircle is less than a right angle.

The angle in a segment less than a semicircle is greater than a right angle.



Let ABCD be a circle, of which BC is a diameter, and E the centre; and let AC be a chord dividing the circle into the segments ABC, ADC, of which the segment ABC is greater, and the segment ADC is less than a semicircle.

Then (i) *the angle in the semicircle* BAC *shall be a right angle*;

(ii) *the angle in the segment* ABC *shall be less than a rt. angle*;

(iii) *the angle in the segment* ADC *shall be greater than a rt. angle.*

Construction. In the arc ADC take any point D;
Join BA, AD, DC, AE; and produce BA to F.

Proof. (i) Because EA = EB, I. *Def.* 15.
∴ the ∠ EAB = the ∠ EBA. I. 5.
And because EA = EC,
∴ the ∠ EAC = the ∠ ECA.
∴ the whole ∠ BAC = the sum of the ∠s EBA, ECA:
but the ext. ∠ FAC = the sum of the two int. ∠s CBA, BCA;
∴ the ∠ BAC = the ∠ FAC;
∴ these angles, being adjacent, are rt. angles.
∴ the ∠ BAC, in the semicircle BAC, is a rt. angle.

(ii) In the △ ABC, because the sum of the ∠ˢ ABC, BAC is less than two rt. angles; I. 17.
and of these, the ∠ BAC is a rt. angle; *Proved.*
∴ the ∠ ABC, which is the angle in the segment ABC, is less than a rt. angle.

(iii) Because ABCD is a quadrilateral inscribed in the ⊙ ABC,
∴ the opp. ∠ˢ ABC, ADC together = two rt. angles; III. 22.
and of these, the ∠ ABC is less than a rt. angle: *Proved.*
∴ the ∠ ADC, which is the angle in the segment ADC, is greater than a rt. angle. Q.E.D.

EXERCISES.

1. *A circle described on the hypotenuse of a right-angled triangle as diameter, passes through the opposite angular point.*

2. A system of right-angled triangles is described upon a given straight line as hypotenuse; find the locus of the opposite angular points.

3. A straight rod of given length slides between two straight rulers placed at right angles to one another; find the locus of its middle point.

4. Two circles intersect at A and B; and through A two diameters AP, AQ are drawn, one in each circle: shew that the points P, B, Q are collinear. [See Def. p. 110.]

5. A circle is described on one of the equal sides of an isosceles triangle as diameter. Shew that it passes through the middle point of the base.

6. Of two circles which have internal contact, the diameter of the inner is equal to the radius of the outer. Shew that any chord of the outer circle, drawn from the point of contact, is bisected by the circumference of the inner circle.

7. Circles described on any two sides of a triangle as diameters intersect on the third side, or the third side produced.

8. *Find the locus of the middle points of chords of a circle drawn through a fixed point. Distinguish between the cases when the given point is within, on, or without the circumference.*

9. Describe a square equal to the difference of two given squares.

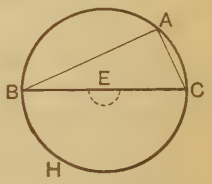
10. Through one of the points of intersection of two circles draw a chord of one circle which shall be bisected by the other.

11. On a given straight line as base a system of equilateral four-sided figures is described: find the locus of the intersection of their diagonals.

NOTES ON PROPOSITION 31.

NOTE 1. The extension of Proposition 20 to *straight* and *reflex* angles furnishes a simple alternative proof of the first theorem contained in Proposition 31, namely,

The angle in a semicircle is a right angle.



For, in the adjoining figure, the angle at the centre, standing on the arc BHC, is double the angle BAC at the $\bigcirc^{ce}$, standing on the same arc.

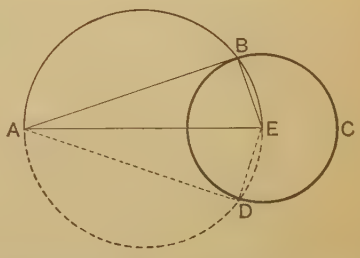
Now the angle at the centre is the *straight angle* BEC ;
∴ the ∠ BAC is half of the *straight angle* BEC :
and a straight angle = two rt. angles ;
∴ the ∠ BAC = one half of two rt. angles,
= one rt. angle. Q.E.D.

NOTE 2. From Proposition 31 we may derive a simple practical solution of Proposition 17, namely,

To draw a tangent to a circle from a given external point.

Let BCD be the given circle, and A the given external point.

It is required to draw from A *a tangent to the* ⊙ BCD.



Find E, the centre of the given circle, and join AE.

On AE describe the semicircle ABE, to cut the given circle at B.

Join AB.

Then AB *shall be a tangent to the* ⊙ BCD.

For the ∠ ABE, being in a semicircle, is a rt. angle. III. 31.

∴ AB is drawn at rt. angles to the radius EB, from its extremity B ;

∴ AB is a tangent to the circle. III. 16.

Q.E.F.

Since the semicircle might be described on either side of AE, it is clear that there will be a second solution of the problem, as shewn by the dotted lines of the figure.

QUESTIONS FOR REVISION AND NUMERICAL EXERCISES.

1. Define an *arc*, a *chord*, a *segment* of a circle. When are segments of circles said to be *similar* to one another?

2. Enunciate propositions which give the properties of *chords* of a circle in relation to the *centre*.

3. Prove that in a circle whose diameter is 34 inches, a chord 30 inches in length is at a distance of 8 inches from the centre.

4. In a circle a chord 2 feet in length stands at a distance of 5 inches from the centre: shew that the diameter of the circle is 2 inches longer than the chord.

5. What must be the length of a chord which is 1 foot distant from the centre of a circle, if the diameter is 2 yards 2 inches?

6. Two parallel chords of a circle, whose diameter is 13 inches, are respectively 5 inches and 1 foot in length: shew that the distance between them is $8\frac{1}{2}$ inches, or $3\frac{1}{2}$ inches.

7. Two circles, whose radii are respectively 26 inches and 25 inches, intersect at two points which are 4 feet apart. Shew that the distance between their centres is 17 inches.

8. The diameters of two concentric circles are respectively 50 inches and 48 inches: shew that any chord of the outer circle which touches the inner must be 14 inches in length.

9. Of two concentric circles the diameter of the greater is 74 inches, and any chord of it which touches the smaller circle is 70 inches in length: shew that the diameter of the smaller circle is 2 feet.

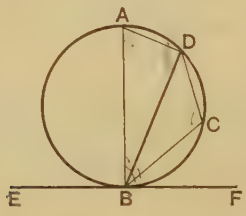
10. Two circles of diameters 74 and 40 inches respectively have a common chord 2 feet in length: shew that the distance between their centres is 51 inches.

11. The chord of an arc is 24 inches in length, and the height of the arc is 8 inches; shew that the diameter of the circle is 26 inches.

12. AB is a line 20 inches in length, and C is its middle point. On AB, AC, CB semicircles are described. Shew that if a circle is inscribed in the space enclosed by the three semicircles its radius must be $3\frac{1}{3}$ inches.

Proposition 32. Theorem.

If a straight line touches a circle, and from the point of contact a chord is drawn, the angles which this chord makes with the tangent shall be equal to the angles in the alternate segments of the circle.



Let EF touch the given ⊙ ABC at B, and let BD be a chord drawn from B, the point of contact.

Then shall

(i) *the* ∠ DBF = *the angle in the alternate segment* BAD :

(ii) *the* ∠ DBE = *the angle in the alternate segment* BCD.

Construction. From B draw BA perp. to EF. I. 11.
Take any point C in the arc BD ;
and join AD, DC, CB.

(i) **Proof.** Because BA is drawn perp. to the tangent EF, at its point of contact B,

∴ BA passes through the centre of the circle : III. 19.

∴ the ∠ ADB, being in a semicircle, is a rt. angle : III. 31.

∴ in the △ ABD, the other ∠s ABD, BAD together = a rt. angle ; I. 32.

that is, the ∠s ABD, BAD together = the ∠ ABF.

From these equals take the common ∠ ABD ;

∴ the ∠ DBF = the ∠ BAD, which is in the alternate segment.

(ii) Because ABCD is a quadrilateral inscribed in a circle,

$\therefore$ the opp. $\angle^s$ BCD, BAD together = two rt. angles : III. 22.

but the $\angle^s$ DBE, DBF together = two rt. angles ; I. 13.

$\therefore$ the $\angle^s$ DBE, DBF together = the $\angle^s$ BCD, BAD ;

and of these the $\angle$ DBF = the $\angle$ BAD ; *Proved.*

$\therefore$ the $\angle$ DBE = the $\angle$ BCD, which is in the alternate segment.

Q.E.D.

EXERCISES.

1. State and prove the converse of Proposition 32.

2. Use this proposition to shew that the tangents drawn to a circle from an external point are equal.

3. If two circles touch one another, any straight line drawn through the point of contact cuts off similar segments.

Prove this for (i) internal, (ii) external contact.

4. If two circles touch one another, and from A, the point of contact, two chords APQ, AXY are drawn : then PX and QY are parallel.

Prove this for (i) internal, (ii) external contact.

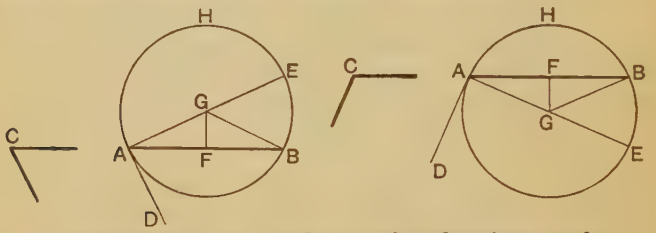
5. Two circles intersect at the points A, B : and one of them passes through O, the centre of the other : prove that OA bisects the angle between the common chord and the tangent to the first circle at A.

6. Two circles intersect at A and B ; and through P, any point on the circumference of one of them, straight lines PAC, PBD are drawn to cut the other circle at C and D : shew that CD is parallel to the tangent at P.

7. If from the point of contact of a tangent to a circle, a chord is drawn, the perpendiculars dropped on the tangent and chord from the middle point of either arc cut off by the chord are equal.

Proposition 33. Problem.

On a given straight line to describe a segment of a circle which shall contain an angle equal to a given angle.



Let AB be the given st. line, and C the given angle.

It is required to describe on AB *a segment of a circle which shall contain an angle equal to* C.

Construction.

At A in BA, make the ∠ BAD equal to the ∠ C. I. 23.

From A draw AE at rt. angles to AD. I. 11.

Bisect AB at F. I. 10.

From F draw FG at rt. angles to AB, cutting AE at G.

Join GB.

Then in the △ˢ AFG, BFG,

Because { AF = BF, *Constr.*
and FG is common,
and the ∠ AFG = the ∠ BFG, being rt. angles ;

∴ GA = GB : I. 4.

∴ the circle described with centre G, and radius GA, will pass through B.

Describe this circle, and call it ABH.

Then the segment AHB *shall contain an angle equal to* C.

Proof. Because AD is drawn at rt. angles to the radius GA from its extremity A,

∴ AD is a tangent to the circle ; III. 16.

and from A, its point of contact, a chord AB is drawn ;

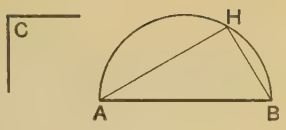
∴ the ∠ BAD = the angle in the alt. segment AHB. III. 32.

But the ∠ BAD = the ∠ C : *Constr.*

∴ the angle in the segment AHB = the ∠ C.

∴ AHB is the segment required. Q.E.F.

NOTE. In the particular case when the given angle C is a rt. angle, the segment required will be the semicircle described on the given st. line AB; for the angle in a semicircle is a rt. angle. III. 31.



EXERCISES.

[The following exercises depend on the corollary to the Converse of Proposition 21 given on page 201, namely

The locus of the vertices of triangles which stand on the same base and have a given vertical angle, is the arc of the segment standing on this base, and containing an angle equal to the given angle.

Exercises 1 and 2 afford good illustrations of the method of finding required points by the *Intersection of Loci*. See page 125.]

1. *Describe a triangle on a given base, having a given vertical angle, and having its vertex on a given straight line.*

2. *Construct a triangle, having given the base, the vertical angle and*
   (i) *one other side.*
   (ii) *the altitude.*
   (iii) *the length of the median which bisects the base.*
   (iv) *the point at which the perpendicular from the vertex meets the base.*

3. *Construct a triangle having given the base, the vertical angle, and the point at which the base is cut by the bisector of the vertical angle.*

[Let AB be the base, X the given point in it, and K the given angle. On AB describe a segment of a circle containing an angle equal to K; complete the $\bigcirc^{ce}$ by drawing the arc APB. Bisect the arc APB at P: join PX, and produce it to meet the $\bigcirc^{ce}$ at C. Then ABC shall be the required triangle.]

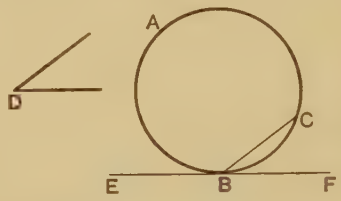
4. *Construct a triangle having given the base, the vertical angle, and the sum of the remaining sides.*

[Let AB be the given base, K the given angle, and H the given line equal to the sum of the sides. On AB describe a segment containing an angle equal to K, also another segment containing an angle equal to half the $\angle$ K. From centre A, with radius H, describe a circle cutting the arc of the last drawn segment at X and Y. Join AX (or AY) cutting the arc of the first segment at C. Then ABC shall be the required triangle.]

5. Construct a triangle having given the base, the vertical angle, and the difference of the remaining sides.

Proposition 34. Problem.

From a given circle to cut off a segment which shall contain an angle equal to a given angle.



Let ABC be the given circle, and D the given angle.

It is required to cut off from the ⊙ ABC *a segment which shall contain an angle equal to* D.

Construction. Take any point B on the ○ce,
and at B draw the tangent EBF. III. 17.
At B, in FB, make the ∠ FBC equal to the ∠ D. I. 23.
Then the segment BAC *shall contain an angle equal to* D.

Proof. Because EF is a tangent to the circle, and from B, its point of contact, a chord BC is drawn,
∴ the ∠ FBC = the angle in the alternate segment BAC.
III. 32.
But the ∠ FBC = the ∠ D; *Constr.*
∴ the angle in the segment BAC = the ∠ D.
Hence from the given ⊙ ABC a segment BAC has been cut off, containing an angle equal to D. Q.E.F.

EXERCISES.

1. The chord of a given segment of a circle is produced to a fixed point: on this straight line so produced draw a segment of a circle similar to the given segment.

2. Through a given point without a circle draw a straight line that will cut off a segment capable of containing an angle equal to a given angle.

QUESTIONS FOR REVISION.

1. Enunciate the propositions from which we infer that a straight line and a circle must either

(i) intersect in two points ; or
(ii) touch at one point ; or
(iii) have no point in common.

2. Give two independent constructions for drawing a tangent to a circle from an external point.

Shew that the two tangents so drawn

(i) are equal ;
(ii) subtend equal angles at the centre ;
(iii) make equal angles with the straight line which joins the given point to the centre.

3. Enunciate propositions relating to

(i) angles in a segment of a circle ;
(ii) similar segments of circles.

4. What are *conjugate arcs* of a circle ?

The angles in conjugate segments of a circle are supplementary. How does Euclid enunciate this theorem ? State and prove its converse.

5. Explain what is meant by a *reflex angle*. What simplifications may be made in the proofs of Third Book Propositions if reflex angles are admitted ?

6. If the circumference of a circle is divided into six equal arcs, shew that the chords joining successive points of division are all equal to the radius of the circle.

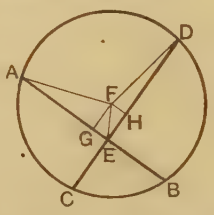
7. Find the locus of the centres of all circles

(i) which pass through two given points ;
(ii) which touch a given circle at a given point ;
(iii) which are of given radius, and touch a given circle ;
(iv) which are of given radius, and pass through a given point ;
(v) which touch a given straight line at a given point ;
(vi) which touch each of two parallel straight lines ;
(vii) which touch each of two intersecting straight lines of unlimited length.

8. If a system of triangles stand on the same base and on the same side of it, and have equal vertical angles, shew that the locus of their vertices is the arc of a circle. Prove this theorem, having first enunciated the proposition of which it is the converse.

Proposition 35. Theorem.

If two chords of a circle cut one another, the rectangle contained by the segments of one shall be equal to the rectangle contained by the segments of the other.



Let AB, CD, two chords of the ⊙ ACBD, cut one another at E.

Then shall the rect. AE, EB = *the rect.* CE, ED.

Construction. Find F, the centre of the ⊙ ACB; III. 1.
From F draw FG, FH perp. respectively to AB, CD. I. 12.
Join FA, FE, FD.

Proof. Because FG is drawn from the centre F perp. to AB,
∴ AB is bisected at G. III. 3.
For a similar reason CD is bisected at H.
Again, because AB is divided equally at G, and unequally at E,
∴ the rect. AE, EB with the sq. on EG = the sq. on AG. II. 5.
To each of these equals add the sq. on GF;
then the rect. AE, EB with the sqq. on EG, GF = the sum of the sqq. on AG, GF.
But the sqq. on EG, GF = the sq. on FE; I. 47.
and the sqq. on AG, GF = the sq. on AF;
for the angles at G are rt. angles.
∴ the rect. AE, EB with the sq. on FE = the sq. on AF.
Similarly it may be shewn that
the rect. CE, ED with the sq. on FE = the sq. on FD.
But the sq. on AF = the sq. on FD; for AF = FD.
∴ the rect. AE, EB with the sq. on FE = the rect. CE, ED with the sq. on FE.
From these equals take the sq. on FE:
then the rect. AE, EB = the rect. CE, ED. Q.E.D.

COROLLARY. *If through a fixed point within a circle any number of chords are drawn, the rectangles contained by their segments are all equal.*

NOTE. The following special cases of this proposition deserve notice:

(i) when the given chords both pass through the centre:
(ii) when one chord passes through the centre, and cuts the other at right angles:
(iii) when one chord passes through the centre, and cuts the other obliquely.

In each of these cases the general proof requires some modification, which may be left as an exercise to the student.

EXERCISES.

1. *Two straight lines* AB, CD *intersect at* E, *so that the rectangle* AE, EB *is equal to the rectangle* CE, ED; *shew that the four points* A, B, C, D *are concyclic.*

2. The rectangle contained by the segments of any chord drawn through a given point within a circle is equal to the square on half the shortest chord which may be drawn through that point.

3. ABC is a triangle right-angled at C; and from C a perpendicular CD is drawn to the hypotenuse: shew that the square on CD is equal to the rectangle AD, DB.

4. ABC is a triangle; and AP, BQ, the perpendiculars dropped from A and B on the opposite sides, intersect at O: shew that the rectangle AO, OP is equal to the rectangle BO, OQ.

5. Two circles intersect at A and B, and through any point in AB their common chord two chords are drawn, one in each circle; shew that their four extremities are concyclic.

6. A and B are two points within a circle such that the rectangle contained by the segments of any chord drawn through A is equal to the rectangle contained by the segments of any chord through B: shew that A and B are equidistant from the centre.

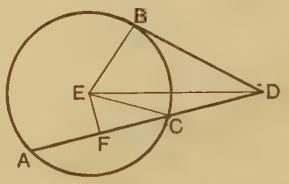
7. *If through* E, *a point without a circle, two secants,* EAB, ECD *are drawn; shew that the rectangle* EA, EB *is equal to the rectangle* EC, ED.

[Proceed as in III. 35, using II. 6.]

8. Through A, a point of intersection of two circles, two straight lines CAE, DAF are drawn, each passing through a centre and terminated by the circumferences: shew that the rectangle CA, AE is equal to the rectangle DA, AF.

Proposition 36. Theorem.

If from any point without a circle a tangent and a secant are drawn, then the rectangle contained by the whole secant and the part of it without the circle shall be equal to the square on the tangent.



Let ABC be a circle; and from D, a point without it, let there be drawn the secant DCA, and the tangent DB.

Then the rect. DA, DC *shall be equal to the sq. on* DB.

Construction. Find E, the centre of the ⊙ ABC: III. 1.
and from E, draw EF perp. to AD. I. 12.
Join EB, EC, ED.

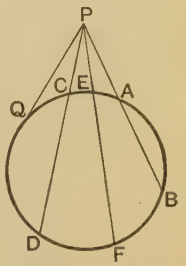
Proof. Because EF, passing through the centre, is perp. to the chord AC,
∴ AC is bisected at F. III. 3.

And since AC is bisected at F and produced to D,
∴ the rect. DA, DC with the sq. on FC = the sq. on FD. II. 6.
To each of these equals add the sq. on EF:
then the rect. DA, DC with the sqq. on EF, FC = the sqq. on EF, FD.
But the sqq. on EF, FC = the sq. on EC; for EFC is a rt. angle;
= the sq. on EB.
And the sqq. on EF, FD = the sq. on ED; for EFD is a rt. angle;
= the sqq. on EB, BD; for EBD is a rt. angle. III. 18.
∴ the rect. DA, DC with the sq. on EB = the sqq. on EB, BD.
From these equals take the sq. on EB:
then the rect. DA, DC = the sq. on DB. Q.E.D.

Note. This proof may easily be adapted to the case when the secant passes through the centre of the circle.

COROLLARY. *If from a given point without a circle any number of secants are drawn, the rectangles contained by the whole secants and the parts of them without the circle are all equal; for each of these rectangles is equal to the square on the tangent drawn from the given point to the circle.*

For instance, in the adjoining figure, each of the rectangles PB, PA and PD, PC and PF, PE is equal to the square on the tangent PQ:



$\therefore$ the rect. PB, PA

= the rect. PD, PC

= the rect. PF, PE.

NOTE. Remembering that the segments into which the chord AB is divided at P, are the lines PA, PB, (see Def., page 139) we are enabled to include the corollaries of Propositions 35 and 36 in a single enunciation.

If any number of chords of a circle are drawn through a given point within or without a circle, the rectangles contained by the segments *of the chords are equal.*

EXERCISES.

1. Use this proposition to shew that tangents drawn to a circle from an external point are equal.

2. If two circles intersect, tangents drawn to them from any point in their common chord produced are equal.

3. If two circles intersect at A and B, and PQ is a tangent to both circles; shew that AB produced bisects PQ.

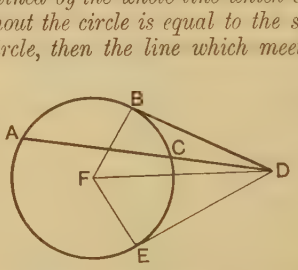
4. If P is any point on the straight line AB produced, shew that the tangents drawn from P to all circles which pass through A and B are equal.

5. ABC is a triangle right-angled at C, and from any point P in AC, a perpendicular PQ is drawn to the hypotenuse: shew that the rectangle AC, AP is equal to the rectangle AB, AQ.

6. ABC is a triangle right-angled at C, and from C a perpendicular CD is drawn to the hypotenuse: shew that the rect. AB, AD is equal to the square on AC.

Proposition 37. Theorem.

If from a point without a circle there are drawn two straight lines, one of which cuts the circle, and the other meets it, and if the rectangle contained by the whole line which cuts the circle and the part of it without the circle is equal to the square on the line which meets the circle, then the line which meets the circle shall be a tangent to it.



Let ABC be a circle; and from D, a point without it, let there be drawn two st lines DCA and DB, of which DCA cuts the circle at C and A, and DB meets it; and let the rect. DA, DC = the sq. on DB.

Then shall DB *be a tangent to the circle.*

Construction. From D draw DE to touch the ⊙ ABC: III. 17.
let E be the point of contact.
Find the centre F, and join FB, FD, FE. III. 1.

Proof. Since DCA is a secant, and DE a tangent to the circle,
∴ the rect. DA, DC = the sq. on DE, III. 36.
But, *by hypothesis*, the rect. DA, DC = the sq. on DB;
∴ the sq. on DE = the sq. on DB;
∴ DE = DB.

Hence in the △ˢ DBF, DEF,

Because { DB = DE, *Proved.*
and BF = EF; I. *Def.* 15.
and DF is common;
∴ the ∠ DBF = the ∠ DEF. I. 8.

But DEF is a rt. angle, for DE is a tangent; III. 18.
∴ DBF is also a rt. angle;
and since BF is a radius,
∴ DB touches the ⊙ ABC at the point B. Q.E.D.

NOTE ON THE METHOD OF LIMITS AS APPLIED TO TANGENCY.

Euclid defines a tangent to a circle as *a straight line which meets the circumference, but being produced, does not cut it:* and from this definition he deduces the fundamental theorem that a tangent is perpendicular to the radius drawn to the point of contact. III. Prop. 16.

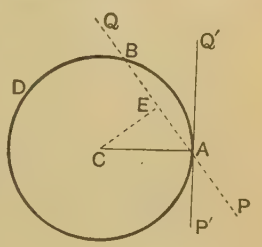
But this result may also be established by the Method of Limits, which regards the tangent as *the ultimate position of a secant when its two points of intersection with the circumference are brought into coincidence* [See Note on page 165]: and it may be shewn that every theorem relating to the tangent may be derived from some more general proposition relating to the secant, by considering the ultimate case when the two points of intersection coincide.

1. *To prove by the Method of Limits that a tangent to a circle is at right angles to the radius drawn to the point of contact.*

Let ABD be a circle, whose centre is C; and PABQ a secant cutting the ○ce in A and B; and let P′AQ′ be the limiting position of PQ when the point B is brought into coincidence with A.

Then shall CA *be perp. to* P′Q′.

Bisect AB at E and join CE:
then CE is perp. to PQ. III. 3.



Now let the secant PABQ change its position in such a way that while the point A remains fixed, the point B continually approaches A, and ultimately coincides with it;

then, *however near* B *approaches to* A, the st. line CE is always perp. to PQ, since it joins the centre to the middle point of the chord AB.

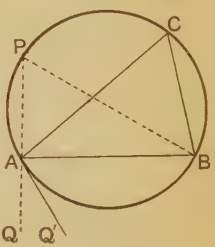
But in the limiting position, when B coincides with A, and the secant PQ becomes the tangent P′Q′, it is clear that the point E will also coincide with A; and the perpendicular CE becomes the radius CA. Hence CA is perp. to the tangent P′Q′ at its point of contact A. Q.E.D.

NOTE. It follows from Proposition 2 that *a straight line cannot cut the circumference of a circle at more than two points.* Now when the two points in which a secant cuts a circle move towards coincidence, the secant ultimately becomes a tangent to the circle: we infer therefore that a tangent cannot meet a circle otherwise than at its point of contact. Thus Euclid's definition of a tangent may be deduced from that given by the Method of Limits.

2. *By this method Proposition* 32 *may be derived as a special case from Proposition* 21.

For let A and B be two points on the $\bigcirc^{ce}$ of the $\odot$ ABC;
and let BCA, BPA be any two angles in the segment BCPA:
then the $\angle$ BPA = the $\angle$ BCA. III. 21.

Produce PA to Q.



Now let the point P continually approach the fixed point A, and ultimately coincide with it;

then, *however near* P *may approach to* A,
the $\angle$ BPQ = the $\angle$ BCA. III. 21.

But in the limiting position when P coincides with A,

and the secant PAQ becomes the tangent AQ',
it is clear that BP will coincide with BA,
and the $\angle$ BPQ becomes the $\angle$ BAQ'.

Hence the $\angle$ BAQ' = the $\angle$ BCA, in the alternate segment. Q.E.D.

The contact of circles may be treated in a similar manner by adopting the following definition.

DEFINITION. If one or other of two intersecting circles alters its position in such a way that the two points of intersection continually approach one another, and ultimately coincide; in the limiting position they are said to **touch** one another, and the point in which the two points of intersection ultimately coincide is called the **point of contact.**

EXAMPLES ON LIMITS.

1. Deduce Proposition 19 from the Corollary of Proposition 1 and Proposition 3.

2. Deduce Propositions 11 and 12 from Ex. 1, page 171.

3. Deduce Proposition 6 from Proposition 5.

4. Deduce Proposition 13 from Proposition 10.

5. Shew that a straight line cuts a circle in two different points, two coincident points, or not at all, according as its distance from the centre is less than, equal to, or greater than a radius.

6. Deduce Proposition 32 from Ex. 3, page 202.

7. Deduce Proposition 36 from Ex. 7, page 227.

8. *The angle in a semi-circle is a right angle.*

To what Theorem is this statement reduced, when the vertex of the right angle is brought into coincidence with an extremity of the diameter?

9. From Ex. 1, page 204, deduce the corresponding property of a triangle inscribed in a circle.

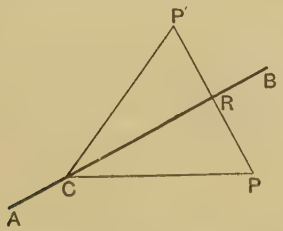
THEOREMS AND EXAMPLES ON BOOK III.

I. ON THE CENTRE AND CHORDS OF A CIRCLE.

[See Propositions 1, 3, 14, 15, 25.]

1. *All circles which pass through a fixed point, and have their centres on a given straight line, pass also through a second fixed point.*

Let AB be the given st. line, and P the given point.



From P draw PR perp. to AB;
and produce PR to P′, making RP′ equal to PR.

Then all circles which pass through P, *and have their centres on* AB, *shall pass also through* P′.

For let C be the centre of *any one* of these circles.
Join CP, CP′.

Then in the $\triangle^s$ CRP, CRP′,

Because { CR is common,
and RP = RP′, *Constr.*
and the $\angle$ CRP = the $\angle$ CRP′, being rt. angles;
$\therefore$ CP = CP′; I. 4.

$\therefore$ the circle whose centre is C, and which passes through P, must pass also through P′.

But C is the centre of *any* circle of the system;

$\therefore$ all circles, which pass through P, and have their centres in AB, pass also through P′. Q. E. D.

2. *Describe a circle that shall pass through three given points not in the same straight line.*

3. Describe a circle that shall pass through two given points and have its centre in a given straight line. When is this impossible?

4. Describe a circle of given radius to pass through two given points. When is this impossible?

5. ABC is an isosceles triangle; and from the vertex A, as centre, a circle is described cutting the base, or the base produced, at X and Y. Shew that BX=CY.

6. If two circles which intersect are cut by a straight line parallel to the common chord, shew that the parts of it intercepted between the circumferences are equal.

7. If two circles cut one another, any two straight lines drawn through a point of section, making equal angles with the common chord, and terminated by the circumferences, are equal.

[Ex. 12, p. 171.]

8. If two circles cut one another, of all straight lines drawn through a point of section and terminated by the circumferences, the greatest is that which is parallel to the line joining the centres.

9. Two circles, whose centres are C and D, intersect at A, B; and through A a straight line PAQ is drawn terminated by the circumferences: if PC, QD intersect at X, shew that the angle PXQ is equal to the angle CAD.

10. Through a point of section of two circles which cut one another draw a straight line terminated by the circumferences and bisected at the point of section.

11. AB is a fixed diameter of a circle, whose centre is C; and from P, any point on the circumference, PQ is drawn perpendicular to AB; shew that the bisector of the angle CPQ always intersects the circle in one or other of two fixed points.

12. Circles are described on the sides of a quadrilateral as diameters: shew that the common chord of any two consecutive circles is parallel to the common chord of the other two.

[Ex. 9, p. 105.]

13. Two equal circles touch one another externally, and through the point of contact two chords are drawn, one in each circle, at right angles to each other: shew that the straight line joining their other extremities is equal to the diameter of either circle.

14. Straight lines are drawn from a given external point to the circumference of a circle: find the locus of their middle points. [Ex. 3, p. 105.]

15. Two equal segments of circles are described on opposite sides of the same chord AB; and through O, the middle point of AB, any straight line POQ is drawn, intersecting the arcs of the segments at P and Q: shew that OP=OQ.

II. ON THE TANGENT AND THE CONTACT OF CIRCLES.

[See Propositions 11, 12, 16, 17, 18, 19.]

1. All equal chords placed in a given circle touch a fixed concentric circle.

2. If from an external point two tangents are drawn to a circle, the angle contained by them is double the angle contained by the chord of contact and the diameter drawn through one of the points of contact.

3. Two circles touch one another externally, and through the point of contact a straight line is drawn terminated by the circumferences: shew that the tangents at its extremities are parallel.

4. Two circles intersect, and through one point of section any straight line is drawn terminated by the circumferences: shew that the angle between the tangents at its extremities is equal to the angle between the tangents at the point of section.

5. Shew that two parallel tangents to a circle intercept on any third tangent a segment which subtends a right angle at the centre.

6. Two tangents are drawn to a given circle from a fixed external point A, and any third tangent cuts them produced at P and Q: shew that PQ subtends a constant angle at the centre of the circle.

7. *In any quadrilateral circumscribed about a circle, the sum of one pair of opposite sides is equal to the sum of the other pair.*

8. *If the sum of one pair of opposite sides of a quadrilateral is equal to the sum of the other pair, shew that a circle may be inscribed in the figure.*

[Bisect two adjacent angles of the figure, and so describe a circle to touch three of its sides. Then prove indirectly by means of the last exercise that this circle must also touch the fourth side.]

9. Two circles touch one another internally, the centre of the outer being within the inner circle: shew that of all chords of the outer circle which touch the inner, the greatest is that which is perpendicular to the straight line joining the centres.

10. In any triangle, if a circle is described from the middle point of one side as centre and with a radius equal to half the sum of the other two sides, it will touch the circles described on these sides as diameters.

11. Through a given point, draw a straight line to cut a circle, so that the part intercepted by the circumference may be equal to a given straight line.

In order that the problem may be possible, between what limits must the given line lie, when the given point is (i) without the circle, (ii) within it?

12. A series of circles touch a given straight line at a given point: shew that the tangents to them at the points where they cut a given parallel straight line all touch a fixed circle, whose centre is the given point.

13. If two circles touch one another internally, and any third circle be described touching one internally and the other externally; then the sum of the distances of the centre of this third circle from the centres of the two given circles is constant.

14. Find the locus of points such that the pairs of tangents drawn from them to a given circle contain a constant angle.

15. Find a point such that the tangents drawn from it to two given circles may be equal to two given straight lines. When is this impossible?

16. If three circles touch one another two and two; prove that the tangents drawn to them at the three points of contact are concurrent and equal.

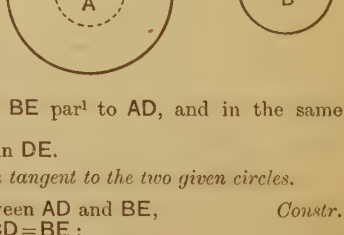
The Common Tangents to Two Circles.

17. *To draw a common tangent to two circles.*

First. When the given circles are external to one another, or when they intersect.

Let A be the centre of the greater circle, and B the centre of the less.

From A, with radius equal to the diff^{ce} of the radii of the given circles, describe a circle: and from B draw BC to touch the last drawn circle. Join AC, and produce it to meet the greater of the given circles at D.



Through B draw the radius BE par^{l} to AD, and in the same direction.

Join DE.

Then DE *shall be a common tangent to the two given circles.*

For since AC = the diff^{ce} between AD and BE, *Constr.*

$\therefore$ CD = BE:

and CD is par^{l} to BE; *Constr.*

$\therefore$ DE is equal and par^{l} to CB. I. 33.

But since BC is a tangent to the circle at C,

$\therefore$ the $\angle$ ACB is a rt. angle; III. 18.

hence each of the angles at D and E is a rt. angle: I. 29.

$\therefore$ DE is a tangent to both circles. Q.E.F.

It follows from hypothesis that the point B is outside the circle used in the construction :

$\therefore$ two tangents such as BC may always be drawn to it from B ; hence *two* common tangents may always be drawn to the given circles by the above method. These are called the **direct common tangents.**

Secondly. When the given circles are external to one another and do not intersect, two more common tangents may be drawn.

For, from centre A, with a radius equal to the *sum* of the radii of the given circles, describe a circle.

From B draw a tangent to this circle ;

and proceed as before, but draw BE in the direction *opposite* to AD.

It follows from hypothesis that B is external to the circle used in the construction ;

$\therefore$ two tangents may be drawn to it from B.

Hence *two* more common tangents may be drawn to the given circles : these will be found to pass between the given circles, and are called the **transverse common tangents.**

Thus, in general, *four* common tangents may be drawn to two given circles.

The student should investigate for himself the number of common tangents which may be drawn in the following special cases, noting in each case where the general construction fails, or is modified :—

(i) When the given circles intersect :

(ii) When the given circles have external contact :

(iii) When the given circles have internal contact :

(iv) When one of the given circles is wholly within the other.

18. *Draw the direct common tangents to two equal circles.*

19. If the two direct, or the two transverse, common tangents are drawn to two circles, the parts of the tangents intercepted between the points of contact are equal.

20. If four common tangents are drawn to two circles external to one another ; shew that the two direct, and also the two transverse, tangents intersect on the straight line which joins the centres of the circles.

21. Two given circles have external contact at A, and a direct common tangent is drawn to touch them at P and Q : shew that PQ subtends a right angle at the point A.

22. Two circles have external contact at A, and a direct common tangent is drawn to touch them at P and Q : shew that a circle described on PQ as diameter is touched at A by the straight line which joins the centres of the circles.

23. Two circles whose centres are C and C′ have external contact at A, and a direct common tangent is drawn to touch them at P and Q: shew that the bisectors of the angles PCA, QC′A meet at right angles in PQ. And if R is the point of intersection of the bisectors, shew that RA is also a common tangent to the circles.

24. Two circles have external contact at A, and a direct common tangent is drawn to touch them at P and Q: shew that the square on PQ is equal to the rectangle contained by the diameters of the circles.

25. Draw a tangent to a given circle, so that the part of it intercepted by another given circle may be equal to a given straight line. When is this impossible?

26. Draw a secant to two given circles, so that the parts of it intercepted by the circumferences may be equal to two given straight lines.

Problems on Tangency.

Obs. The following exercises are solved by the Method of Intersection of Loci, explained on page 125.

The student should begin by making himself familiar with the following loci.

(i) *The locus of the centres of circles which pass through two given points.*

(ii) *The locus of the centres of circles which touch a given straight line at a given point.*

(iii) *The locus of the centres of circles which touch a given circle at a given point.*

(iv) *The locus of the centres of circles which touch a given straight line, and have a given radius.*

(v) *The locus of the centres of circles which touch a given circle, and have a given radius.*

(vi) *The locus of the centres of circles which touch two given straight lines.*

In each exercise the student should investigate the limits and relations among the data, in order that the problem may be possible.

27. Describe a circle to touch three given straight lines.

28. Describe a circle to pass through a given point, and touch a given straight line at a given point.

29. Describe a circle to pass through a given point, and touch a given circle at a given point.

30. Describe a circle of given radius to pass through a given point, and touch a given straight line.

31. Describe a circle of given radius to touch two given circles.

32. Describe a circle of given radius to touch two given straight lines.

33. Describe a circle of given radius to touch a given circle and a given straight line.

34. Describe two circles of given radii to touch one another and a given straight line, on the same side of it.

35. If a circle touches a given circle and a given straight line, shew that the points of contact and an extremity of the diameter of the given circle at right angles to the given line are collinear.

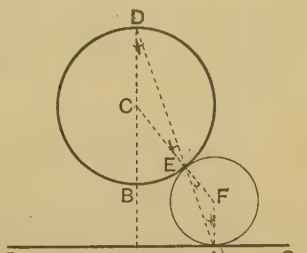
36. *To describe a circle to touch a given circle, and also to touch a given straight line at a given point.*

Let DEB be the given circle, PQ the given straight line, and A the given point in it.

It is required to describe a circle to touch the ⊙ DEB, *and also to touch* PQ *at* A.

At A draw AF perp. to PQ: I. 11. then the centre of the required circle must lie in AF. III. 19.

Find C, the centre of the ⊙ DEB, III. 1. and draw a diameter BD perp. to PQ: join A to one extremity D, cutting the ○ce at E.



Join CE, and produce it to cut AF at F.

Then F *shall be the centre, and* FA *the radius of the required circle.*

[Supply the proof: and shew that a second solution is obtained by joining AB, and producing it to meet the ○ce.
Also distinguish between the nature of the contact of the circles, when PQ cuts, touches, or is without the given circle.]

37. Describe a circle to touch a given straight line, and to touch a given circle at a given point.

38. Describe a circle to touch a given circle, have its centre in a given straight line, and pass through a given point in that straight line.

[For other problems of the same class see page 253.]

Orthogonal Circles.

Definition. Circles which intersect at a point, so that the two tangents at that point are at right angles to one another, are said to be **orthogonal**, or **to cut one another orthogonally**.

39. In two intersecting circles the angle between the tangents at one point of intersection is equal to the angle between the tangents at the other.

40. *If two circles cut one another orthogonally, the tangent to each circle at a point of intersection will pass through the centre of the other circle.*

41. *If two circles cut one another orthogonally, the square on the distance between their centres is equal to the sum of the squares on their radii.*

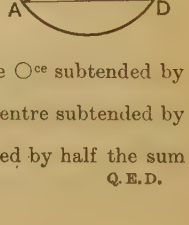
42. Find the locus of the centres of all circles which cut a given circle orthogonally at a given point.

43. Describe a circle to pass through a given point and cut a given circle orthogonally at a given point.

III. ON ANGLES IN SEGMENTS, AND ANGLES AT THE CENTRES AND CIRCUMFERENCES OF CIRCLES.

[See Propositions 20, 21, 22 ; 26, 27, 28, 29 ; 31, 32, 33, 34.]

1. *If two chords intersect within a circle, they form an angle equal to that at the centre, subtended by half the sum of the arcs they cut off.*



Let AB and CD be two chords, intersecting at E within the given ⊙ ADBC.

Then shall the ∠ AEC *be equal to the angle at the centre, subtended by half the sum of the arcs* AC, BD.

Join AD.

Then the ext. ∠ AEC = the sum of the int. opp. ∠ˢ EDA, EAD ;

that is, the sum of the ∠ˢ CDA, BAD.

But the ∠ˢ CDA, BAD are the angles at the ○ᶜᵉ subtended by the arcs AC, BD ;

∴ their sum = half the sum of the angles at the centre subtended by the same arcs ;

or, the ∠ AEC = the angle at the centre subtended by half the sum of the arcs AC, BD. Q.E.D.

2. *If two chords when produced intersect outside a circle, they form an angle equal to that at the centre subtended by half the difference of the arcs they cut off.*

3. The sum of the arcs cut off by two chords of a circle at right angles to one another is equal to the semi-circumference.

4. AB, AC are any two chords of a circle; and P, Q are the middle points of the minor arcs cut off by them: if PQ is joined, cutting AB and AC at X, Y, shew that AX = AY.

5. *If one side of a quadrilateral inscribed in a circle is produced, the exterior angle is equal to the opposite interior angle.*

6. If two circles intersect, and any straight lines are drawn, one through each point of section, terminated by the circumferences; shew that the chords which join their extremities towards the same parts are parallel.

7. ABCD is a quadrilateral inscribed in a circle; and the opposite sides AB, DC are produced to meet at P, and CB, DA to meet at Q: if the circles circumscribed about the triangles PBC, QAB intersect at R, shew that the points P, R, Q are collinear.

8. If a circle is described on one of the sides of a right-angled triangle, then the tangent drawn to it at the point where it cuts the hypotenuse bisects the other side.

9. Given three points not in the same straight line: shew how to find any number of points on the circle which passes through them, without finding the centre.

10. Through any one of three given points not in the same straight line, draw a tangent to the circle which passes through them, without finding the centre.

11. Of two circles which intersect at A and B, the circumference of one passes through the centre of the other: from A any straight line is drawn to cut the first at C, the second at D; shew that CB = CD.

12. Two tangents AP, AQ are drawn to a circle, and B is the middle point of the arc PQ, convex to A. Shew that PB bisects the angle APQ.

13. Two circles intersect at A and B; and at A tangents are drawn, one to each circle, to meet the circumferences at C and D; if CB, BD are joined, shew that the triangles ABC, DBA are equiangular to one another.

14. Two segments of circles are described on the same chord and on the same side of it; the extremities of the common chord are joined to any point on the arc of the exterior segment: shew that the arc intercepted on the interior segment is constant.

15. If a series of triangles are drawn standing on a fixed base, and having a given vertical angle, show that the bisectors of the vertical angles all pass through a fixed point.

16. ABC is a triangle inscribed in a circle, and E the middle point of the arc subtended by BC on the side remote from A: if through E a diameter ED is drawn, shew that the angle DEA is half the difference of the angles at B and C. [See Ex. 7, p. 109.]

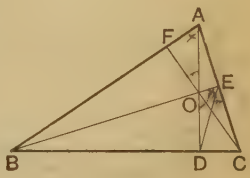
17. If two circles touch each other internally at a point A, any chord of the exterior circle which touches the interior is divided at its point of contact into segments which subtend equal angles at A.

18. If two circles touch one another internally, and a straight line is drawn to cut them, the segments of it intercepted between the circumferences subtend equal angles at the point of contact.

The Orthocentre of a Triangle.

19. *The perpendiculars drawn from the vertices of a triangle to the opposite sides are concurrent.*

In the △ ABC, let AD, BE be the perp^{s} drawn from A and B to the opposite sides; and let them intersect at O. *Join* CO; and produce it to meet AB at F.



It is required to shew that CF *is perp. to* AB.

Join DE.

Then, because the ∠^{s} OEC, ODC are rt. angles, *Hyp.*

∴ the points O, E, C, D are concyclic:

∴ the ∠ DEC = the ∠ DOC, in the same segment;

= the vert. opp. ∠ FOA.

Again, because the ∠^{s} AEB, ADB are rt. angles. *Hyp.*

∴ the points A, E, D, B are concyclic:

∴ the ∠ DEB = the ∠ DAB, in the same segment.

∴ the sum of the ∠^{s} FOA, FAO = the sum of the ∠^{s} DEC, DEB

= a rt. angle: *Hyp.*

∴ the remaining ∠ AFO = a rt. angle: I. 32.

that is, CF is perp. to AB.

Hence the three perp^{s} AD, BE, CF meet at the point O. Q.E.D.

[For an Alternative Proof see p. 114.]

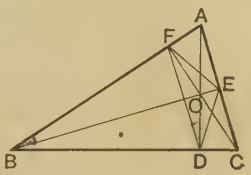
Definitions.

(i) The intersection of the perpendiculars drawn from the vertices of a triangle to the opposite sides is called its **orthocentre.**

(ii) The triangle formed by joining the feet of the perpendiculars is called the **pedal** or **orthocentric triangle.**

20. *In an acute-angled triangle the perpendiculars drawn from the vertices to the opposite sides bisect the angles of the pedal triangle through which they pass.*

In the acute-angled △ ABC, let AD, BE, CF be the perp^{s} drawn from the vertices to the opposite sides, meeting at the orthocentre O ; and let DEF be the pedal triangle.



Then shall AD, BE, CF *bisect respectively the* ∠^{s} FDE, DEF, EFD.

For, as in the last theorem, it may be shewn that the points O, D, C, E are concyclic ;

∴ the ∠ ODE = the ∠ OCE, in the same segment.

Similarly the points O, D, B, F are concyclic ;

∴ the ∠ ODF = the ∠ OBF, in the same segment.

But the ∠ OCE = the ∠ OBF, each being the comp^{t} of the ∠ BAC.

∴ the ∠ ODE = the ∠ ODF.

Similarly it may be shewn that the ∠^{s} DEF, EFD are bisected by BE and CF. Q.E.D.

Corollary. (i) *Every two sides of the pedal triangle are equally inclined to that side of the original triangle in which they meet.*

For the ∠ EDC = the comp^{t} of the ∠ ODE
= the comp^{t} of the ∠ OCE
= the ∠ BAC.

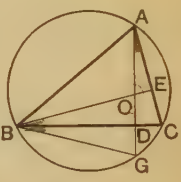
Similarly it may be shewn that the ∠ FDB = the ∠ BAC,
∴ the ∠ EDC = the ∠ FDB = the ∠ A.

In like manner it may be proved that
the ∠ DEC = the ∠ FEA = the ∠ B,
and the ∠ DFB = the ∠ EFA = the ∠ C.

Corollary. (ii) *The triangles* DEC, AEF, DBF *are equiangular to one another and to the triangle* ABC.

Note. If the angle BAC is *obtuse*, then the perpendiculars BE, CF bisect *externally* the corresponding angles of the pedal triangle.

21. *In any triangle, if the perpendiculars drawn from the vertices on the opposite sides are produced to meet the circumscribed circle, then each side bisects that portion of the line perpendicular to it which lies between the orthocentre and the circumference.*



Let ABC be a triangle in which the perpendiculars AD, BE are drawn, intersecting at O the orthocentre, and let AD be produced to meet the $\bigcirc^{ce}$ of the circumscribing circle at G.

Then shall DO = DG.

Join BG.

Then in the two $\triangle^s$ OEA, ODB,
the $\angle$ OEA = the $\angle$ ODB, being rt. angles;
and the $\angle$ EOA = the vert. opp. $\angle$ DOB;
$\therefore$ the remaining $\angle$ EAO = the remaining $\angle$ DBO. I. 32.

But the $\angle$ CAG = the $\angle$ CBG, in the same segment;
$\therefore$ the $\angle$ DBO = the $\angle$ DBG.

Then in the $\triangle^s$ DBO, DBG,

Because { the $\angle$ DBO = the $\angle$ DBG, *Proved.*
the $\angle$ BDO = the $\angle$ BDG,
and BD is common;
$\therefore$ DO = DG. I. 26.

Q. E. D.

22. *In an acute-angled triangle the three sides are the external bisectors of the angles of the pedal triangle: and in an obtuse-angled triangle the sides containing the obtuse angle are the internal bisectors of the corresponding angles of the pedal triangle.*

23. *If* O *is the orthocentre of the triangle* ABC, *shew that the angles* BOC, BAC *are supplementary.*

24. *If* O *is the orthocentre of the triangle* ABC, *then any one of the four points* O, A, B, C *is the orthocentre of the triangle whose vertices are the other three.*

25. *The three circles which pass through two vertices of a triangle and its orthocentre are each equal to the circle circumscribed about the triangle.*

26. D, E are taken on the circumference of a semicircle described on a given straight line AB: the chords AD, BE and AE, BD intersect (produced if necessary) at F and G: shew that FG is perpendicular to AB.

27. ABCD is a parallelogram; AE and CE are drawn at right angles to AB, and CB respectively: shew that ED, if produced, will be perpendicular to AC.

28. ABC is a triangle, O is its orthocentre, and AK a diameter of the circumscribed circle: shew that BOCK is a parallelogram.

29. The orthocentre of a triangle is joined to the middle point of the base, and the joining line is produced to meet the circumscribed circle: prove that it will meet it at the same point as the diameter which passes through the vertex.

30. The perpendicular from the vertex of a triangle on the base, and the straight line joining the orthocentre to the middle point of the base, are produced to meet the circumscribed circle at P and Q: shew that PQ is parallel to the base.

31. *The distance of each vertex of a triangle from the orthocentre is double of the perpendicular drawn from the centre of the circumscribed circle on the opposite side.*

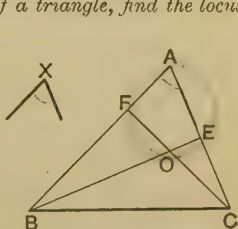
32. Three circles are described each passing through the orthocentre of a triangle and two of its vertices: shew that the triangle formed by joining their centres is equal in all respects to the original triangle.

33. ABC is a triangle inscribed in a circle, and the bisectors of its angles which intersect at O are produced to meet the circumference in PQR: shew that O is the orthocentre of the triangle PQR.

34. Construct a triangle, having given a vertex, the orthocentre, and the centre of the circumscribed circle.

Loci.

35. *Given the base and vertical angle of a triangle, find the locus of its orthocentre.*



Let BC be the given base, and X the given angle; and let BAC be any triangle on the base BC, having its vertical ∠ A equal to the ∠ X.

Draw the perp^{s} BE, CF, intersecting at the orthocentre O.

It is required to find the locus of O.

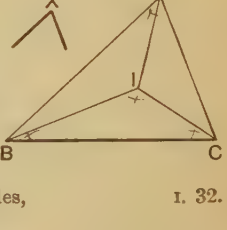
Since the ∠^{s} OFA, OEA are rt. angles,
∴ the points O, F, A, E are concyclic;
∴ the ∠ FOE is the supplement of the ∠ A: III. 22.
∴ the vert. opp. ∠ BOC is the supplement of the ∠ A.

But the ∠ A is constant, being always equal to the ∠ X;
∴ its supplement is constant;

that is, the △ BOC has a fixed base, and constant vertical angle; hence the locus of its vertex O is the arc of a segment of which BC is the chord.

[See Corollary p. 201.]

36. *Given the base and vertical angle of a triangle, find the locus of the intersection of the bisectors of its angles.*



Let BAC be any triangle on the given base BC, having its vertical angle equal to the given ∠ X; and let AI, BI, CI be the bisectors of its angles. [See Ex. 2, p. 111.]

It is required to find the locus of the point I.

Denote the angles of the △ ABC by A, B, C; and let the ∠ BIC be denoted by I.

Then from the △ BIC,

(i) $I+\frac{1}{2}B+\frac{1}{2}C=$ two rt. angles, I. 32.

and from the △ ABC,

$A+B+C=$ two rt. angles; I. 32.

(ii) so that $\frac{1}{2}A+\frac{1}{2}B+\frac{1}{2}C=$ one rt. angle,

∴, taking the differences of the equals in (i) and (ii),

$I-\frac{1}{2}A=$ one rt. angle:

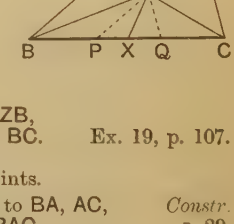
or, $I=$ one rt. angle $+\frac{1}{2}A$.

But A is constant, being always equal to the ∠ X;

∴ I is constant:

∴ the locus of I is the arc of a segment on the fixed chord BC.

37. *Given the base and vertical angle of a triangle, find the locus of the centroid, that is, the intersection of the medians.*



Let BAC be any triangle on the given base BC, having its vertical angle equal to the given angle S; let the medians AX, BY, CZ intersect at the centroid G. [See Ex. 4, p. 113.]

It is required to find the locus of the point G.

Through G draw GP, GQ par^{l} to AB and AC respectively.

Then ZG is a third part of ZC;

Ex. 4, p. 113,

and since GP is par^{l} to ZB,

∴ BP is a third part of BC. Ex. 19, p. 107.

Similarly QC is a third part of BC;

∴ P and Q are fixed points.

Now since PG, GQ are par^{l} respectively to BA, AC, *Constr.*

∴ the ∠ PGQ = the ∠ BAC, I. 29.

= the ∠ S,

that is, the ∠ PGQ is constant;

∴ the locus of G is the arc of a segment on the fixed chord PQ.

NOTE. In this problem the points A and G move on the arcs of *similar segments.*

38. Given the base and the vertical angle of a triangle; find the locus of the intersection of the bisectors of the exterior base angles.

39. Through the extremities of a given straight line AB any two parallel straight lines AP, BQ are drawn; find the locus of the intersection of the bisectors of the angles PAB, QBA.

40. Find the locus of the middle points of chords of a circle drawn through a fixed point.

Distinguish between the cases when the given point is within, on, or without the circumference.

41. Find the locus of the points of contact of tangents drawn from a fixed point to a system of concentric circles.

42. Find the locus of the intersection of straight lines which pass through two fixed points on a circle and intercept on its circumference an arc of constant length.

43. A and B are two fixed points on the circumference of a circle, and PQ is any diameter: find the locus of the intersection of PA and QB.

44. BAC is any triangle described on the fixed base BC and having a constant vertical angle; and BA is produced to P, so that BP is equal to the sum of the sides containing the vertical angle: find the locus of P.

45. AB is a fixed chord of a circle, and AC is a moveable chord passing through A: if the parallelogram CB is completed, find the locus of the intersection of its diagonals.

46. A straight rod PQ slides between two rulers placed at right angles to one another, and from its extremities PX, QX are drawn perpendicular to the rulers: find the locus of X.

47. Two circles whose centres are C and D, intersect at A and B: through A, any straight line PAQ is drawn terminated by the circumferences; and PC, QD intersect at X: find the locus of X, and shew that it passes through B. [Ex. 9, p. 234.]

48. Two circles intersect at A and B, and through P, any point on the circumference of one of them, two straight lines PA, PB are drawn, and produced if necessary, to cut the other circle at X and Y: find the locus of the intersection of AY and BX.

49. Two circles intersect at A and B; HAK is a fixed straight line drawn through A and terminated by the circumferences, and PAQ is any other straight line similarly drawn: find the locus of the intersection of HP and QK.

50. Two segments of circles are on the same chord AB and on the same side of it; and P and Q are any points one on each arc: find the locus of the intersection of the bisectors of the angles PAQ, PBQ.

51. Two circles intersect at A and B; and through A any straight line PAQ is drawn terminated by the circumferences: find the locus of the middle point of PQ.

Miscellaneous Examples on Angles in a Circle.

52. ABC is a triangle, and circles are drawn through B, C, cutting the sides in P, Q, P′, Q′, ...: shew that PQ, P′Q′ ... are parallel to one another and to the tangent drawn at A to the circle circumscribed about the triangle.

53. Two circles intersect at B and C, and from any point A, on the circumference of one of them, AB, AC are drawn, and produced if necessary, to meet the other at D and E: shew that DE is parallel to the tangent at A.

54. A secant PAB and a tangent PT are drawn to a circle from an external point P; and the bisector of the angle ATB meets AB at C: shew that PC is equal to PT.

55. From a point A on the circumference of a circle two chords AB, AC are drawn, and also the diameter AF: if AB, AC are produced to meet the tangent at F in D and E, shew that the triangles ABC, AED are equiangular to one another.

56. O is any point within a triangle ABC, and OD, OE, OF are drawn perpendicular to BC, CA, AB respectively: shew that the angle BOC is equal to the sum of the angles BAC, EDF.

57. If two tangents are drawn to a circle from an external point, shew that they contain an angle equal to the difference of the angles in the segments cut off by the chord of contact.

58. Two circles intersect, and through a point of section a straight line is drawn bisecting the angle between the diameters through that point: shew that this straight line cuts off similar segments from the two circles.

59. Two equal circles intersect at A and B; and from centre A, with any radius less than AB a third circle is described cutting the given circles on the same side of AB at C and D: shew that the points B, C, D are collinear.

60. ABC and A′B′C′ are two triangles inscribed in a circle, so that AB, AC are respectively parallel to A′B′, A′C′: shew that BC′ is parallel to B′C.

61. Two circles intersect at A and B, and through A two straight lines HAK, PAQ are drawn terminated by the circumferences: if HP and KQ intersect at X, shew that the points H, B, K, X are concyclic.

62. Describe a circle touching a given straight line at a given point, so that tangents drawn to it from two fixed points in the given line may be parallel. [See Ex. 10, p. 197.]

63. C is the centre of a circle, and CA, CB two fixed radii: if from any point P on the arc AB perpendiculars PX, PY are drawn to CA and CB, shew that the distance XY is constant.

64. AB is a chord of a circle, and P any point in its circumference; PM is drawn perpendicular to AB, and AN is drawn perpendicular to the tangent at P: shew that MN is parallel to PB.

65. P is any point on the circumference of a circle of which AB is a fixed diameter, and PN is drawn perpendicular to AB; on AN and BN as diameters circles are described, which are cut by AP, BP at X and Y: shew that XY is a common tangent to these circles.

66. Upon the same chord and on the same side of it three segments of circles are described containing respectively a given angle, its supplement and a right angle: shew that the intercept made by the two former segments upon any straight line drawn through an extremity of the given chord is bisected by the latter segment.

67. Two straight lines of indefinite length touch a given circle, and any chord is drawn so as to be bisected by the chord of contact: if the former chord is produced, shew that the intercepts between the circumference and the tangents are equal.

68. Two circles intersect one another: through one of the points of section draw a straight line of given length terminated by the circumferences.

69. On the three sides of any triangle equilateral triangles are described remote from the given triangle: shew that the circles described about them intersect at a point.

70. On BC, CA, AB the sides of a triangle ABC, any points P, Q, R are taken; shew that the circles described about the triangles AQR, BRP, CPQ meet in a point.

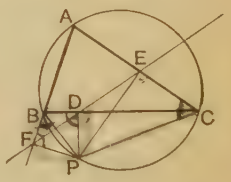
71. Find a point within a triangle at which the sides subtend equal angles.

72. Describe an equilateral triangle so that its sides may pass through three given points.

73. Describe a triangle equal in all respects to a given triangle, and having its sides passing through three given points.

Simson's Line.

74. *If from any point on the circumference of the circle circumscribed about a triangle, perpendiculars are drawn to the three sides, the feet of these perpendiculars are collinear.*



Let P be any point on the $\bigcirc^{ce}$ of the circle circumscribed about the $\triangle$ ABC; and let PD, PE, PF be the perp^{s} drawn from P to the three sides.

It is required to prove that the points D, E, F *are collinear.*

Join FD and DE:
then FD and DE shall be in the same st. line.

Join PB, PC.

Because the $\angle^{s}$ PDB, PFB are rt. angles, *Hyp.*
$\therefore$ the points P, D, B, F are concyclic:
$\therefore$ the $\angle$ PDF = the $\angle$ PBF, in the same segment. III. 21.

But since BACP is a quad^{l} inscribed in a circle, having one of its sides AB produced to F,
$\therefore$ the ext. $\angle$ PBF = the opp. int. $\angle$ ACP. *Ex.* 3, *p.* 202.
$\therefore$ the $\angle$ PDF = the $\angle$ ACP.
To each add the $\angle$ PDE:
then the $\angle^{s}$ PDF, PDE = the $\angle^{s}$ ECP, PDE.
But since the $\angle^{s}$ PDC, PEC are rt. angles,
$\therefore$ the points P, D, E, C are concyclic;
$\therefore$ the $\angle^{s}$ ECP, PDE together = two rt. angles:
$\therefore$ the $\angle^{s}$ PDF, PDE together = two rt. angles;
$\therefore$ FD and DE are in the same st. line; I. 14.
that is, the points D, E, F are collinear. Q.E.D.

[The line FDE is called the **Pedal** or **Simson's Line** of the triangle ABC for the point P; though the tradition attributing the theorem to Robert Simson has been recently shaken by the researches of Dr. J. S. Mackay.]

75. ABC is a triangle inscribed in a circle; and from any point P on the circumference PD, PF are drawn perpendicular to BC and AB: if FD, or FD produced, cuts AC at E, shew that PE is perpendicular to AC.

76. Find the locus of a point which moves so that if perpendiculars are drawn from it to the sides of a given triangle, their feet are collinear.

77. ABC and AB′C′ are two triangles having a common vertical angle, and the circles circumscribed about them meet again at P; shew that the feet of perpendiculars drawn from P to the four lines AB, AC, BC, B′C′ are collinear.

78. *A triangle is inscribed in a circle, and any point* P *on the circumference is joined to the orthocentre of the triangle: shew that this joining line is bisected by the pedal of the point* P.

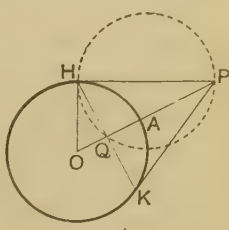
IV. ON THE CIRCLE IN CONNECTION WITH RECTANGLES.

[See Propositions 35, 36, 37.]

1. *If from any external point* P *two tangents are drawn to a given circle whose centre is* O, *and if* OP *meets the chord of contact at* Q; *then the rectangle* OP, OQ *is equal to the square on the radius.*

Let PH, PK be tangents, drawn from the external point P to the ⊙ HAK, whose centre is O; and let OP meet HK the chord of contact at Q, and the ○ce at A.

Then shall the rect. OP, OQ = *the sq. on* OA.



On HP as diameter describe a circle: this circle must pass through Q, since the ∠ HQP is a rt. angle. III. 31.

Join OH.

Then since PH is a tangent to the ⊙ HAK,

∴ the ∠ OHP is a rt. angle.

And since HP is a diameter of the ⊙ HQP,

∴ OH touches the ⊙ HQP at H. III. 16.

∴ the rect. OP, OQ = the sq. on OH, III. 36.

= the sq. on OA. Q.E.D.

2. ABC is a triangle, and AD, BE, CF the perpendiculars drawn from the vertices to the opposite sides, meeting in the orthocentre O: shew that the rect. AO, OD = the rect. BO, OE = the rect. CO, OF.

3. ABC is a triangle, and AD, BE the perpendiculars drawn from A and B on the opposite sides: shew that the rectangle CA, CE is equal to the rectangle CB, CD.

4. ABC is a triangle right-angled at C, and from D, any point in the hypotenuse AB, a straight line DE is drawn perpendicular to AB and meeting BC at E: shew that the square on DE is equal to the difference of the rectangles AD, DB and CE, EB.

5. From an external point P two tangents are drawn to a given circle whose centre is O, and OP meets the chord of contact at Q: shew that any circle which passes through the points P, Q will cut the given circle orthogonally. [See Def. p. 240.]

6. *A series of circles pass through two given points, and from a fixed point in the common chord produced tangents are drawn to all the circles: shew that the points of contact lie on a circle which cuts all the given circles orthogonally.*

7. *All circles which pass through a fixed point, and cut a given circle orthogonally, pass also through a second fixed point.*

8. Find the locus of the centres of all circles which pass through a given point and cut a given circle orthogonally.

9. Describe a circle to pass through two given points and cut a given circle orthogonally.

10. A, B, C, D are four points taken in order on a given straight line: find a point O between B and C such that the rectangle OA, OB may be equal to the rectangle OC, OD.

11. AB *is a fixed diameter of a circle, and* CD *a fixed straight line of indefinite length cutting* AB *or* AB *produced at right angles; any straight line is drawn through* A *to cut* CD *at* P *and the circle at* Q: *shew that the rectangle* AP, AQ *is constant.*

12. AB is a fixed diameter of a circle, and CD a fixed chord at right angles to AB; any straight line is drawn through A to cut CD at P and the circle at Q: shew that the rectangle AP, AQ is equal to the square on AC.

13. A *is a fixed point, and* CD *a fixed straight line of indefinite length*; AP *is any straight line drawn through* A *to meet* CD *at* P; *and in* AP *a point* Q *is taken such that the rectangle* AP, AQ *is constant: find the locus of* Q.

14. Two circles intersect orthogonally, and tangents are drawn from any point on the circumference of one to touch the other: prove that the first circle passes through the middle point of the chord of contact of the tangents. [Ex. 1, p. 251.]

15. A semicircle is described on AB as diameter, and any two chords AC, BD are drawn intersecting at P: shew that

$$AB^2 = AC \cdot AP + BD \cdot BP.$$

16. Two circles intersect at B and C, and the two direct common tangents AE and DF are drawn: if the common chord is produced to meet the tangents at G and H, shew that $GH^2 = AE^2 + BC^2$.

17. If from a point P, without a circle, PM is drawn perpendicular to a diameter AB, and also a secant PCD, shew that

$$PM^2 = PC \cdot PD + AM \cdot MB.$$

18. Three circles intersect at D, and their other points of intersection are A, B, C; AD cuts the circle BDC at E, and EB, EC cut the circles ADB, ADC respectively at F and G: show that the points F, A, G are collinear, and F, B, C, G concyclic.

19. A semicircle is described on a given diameter BC, and from B and C any two chords BE, CF are drawn intersecting within the semicircle at O; BF and CE are produced to meet at A: shew that the sum of the squares on AB, AC is equal to twice the square on the tangent from A together with the square on BC.

20. X and Y are two fixed points in the diameter of a circle equidistant from the centre C: through X any chord PXQ is drawn, and its extremities are joined to Y; shew that the sum of the squares on the sides of the triangle PYQ is constant. [See p. 161, Ex. 24.]

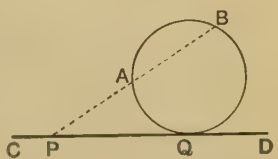
Problems on Tangency.

21. *To describe a circle to pass through two given points and to touch a given straight line.*

Let A and B be the given points, and CD the given st. line.

It is required to describe a circle to pass through A *and* B *and to touch* CD.

Join BA, and produce it to meet CD at P.



Describe a square equal to the rect. PA, PB; II. 14.
and from PD (or PC) cut off PQ equal to a side of this square.

Through A, B, and Q describe a circle. Ex. 4, p. 171.

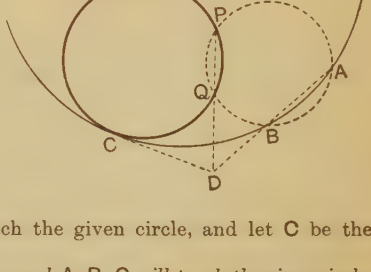
Then since the rect. PA, PB = the sq. on PQ,
∴ the ⊙ ABQ touches CD at Q. III. 37.

Q.E.F.

Notes. (i) Since PQ may be taken on either side of P, it is clear that there are in general two solutions of the problem.

(ii) When AB is parallel to the given line CD, the above method is not applicable. In this case a simple construction follows from III. 1, Cor. and III. 16, and it will be found that only one solution exists.

22. *To describe a circle to pass through two given points and to touch a given circle.*



Let A and B be the given points, and CRP the given circle.

It is required to describe a circle to pass through A *and* B, *and to touch the* ⊙ CRP.

Through A and B describe *any* circle to cut the given circle at P and Q.

Join AB, PQ, and produce them to meet at D.

From D draw DC to touch the given circle, and let C be the point of contact.

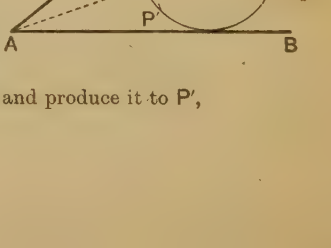
Then the circle described through A, B, C *will touch the given circle.*

For, from the ⊙ ABQP, the rect. DA, DB = the rect. DP, DQ:
and from the ⊙ PQC, the rect. DP, DQ = the sq. on DC; III. 36.
∴ the rect. DA, DB = the sq. on DC:
∴ DC touches the ⊙ ABC at C. III. 37.
But DC touches the ⊙ PQC at C; *Constr.*
∴ the ⊙ ABC touches the given circle, and it passes through the given points A and B. Q.E.F.

NOTE. (i) Since two tangents may be drawn from D to the given circle, it follows that there will be two solutions of the problem.

(ii) The general construction fails when the straight line bisecting AB at right angles passes through the centre of the given circle: the problem then becomes symmetrical, and the solution is obvious.

23. *To describe a circle to pass through a given point and to touch two given straight lines.*



Let P be the given point, and AB, AC the given straight lines.

It is required to describe a circle to pass through P *and to touch* AB, AC.

Now the centre of every circle which touches AB and AC must lie on the bisector of the ∠ BAC. Ex. 7, p. 197.

Hence draw AE bisecting the ∠ BAC.

From P draw PK perp. to AE, and produce it to P', making KP' equal to PK.

Then every circle which has its centre in AE, and passes through P, must also pass through P'. Ex. 1, p. 233.

Hence the problem is now reduced to drawing a circle through P and P' to touch *either* AC or AB. Ex. 21, p. 253.

Produce P'P to meet AC at S.

Describe a square equal to the rect. SP, SP'; II. 14.
and cut off SR equal to a side of the square.
Describe a circle through the points P', P, R.
Then since the rect. SP, SP' = the sq. on SR, *Constr.*
∴ this circle touches AC at R; III. 37,
and since its centre is in AE, the bisector of the ∠ BAC,
it may be shewn also to touch AB. Q.E.F.

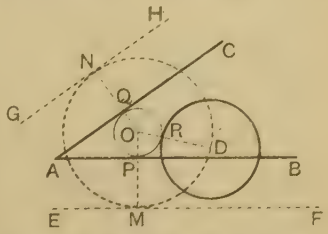
NOTES. (i) Since SR may be taken on either side of S, it follows that there will be two solutions of the problem.

(ii) If the given straight lines are parallel, the centre lies on the parallel straight line mid-way between them, and the construction proceeds as before.

24. *To describe a circle to touch two given straight lines and a given circle.*

Let AB, AC be the two given st. lines, and D the centre of the given circle.

It is required to describe a circle to touch AB, AC *and the circle whose centre is* D.



Draw EF, GH par[l] to AB and AC respectively, on the sides remote from D, and at distances from them equal to the radius of the given circle.

Describe the ⊙ MND to touch EF and GH at M and N, and to pass through D. Ex. 23, p. 254.

Let O be the centre of this circle.

Join OM, ON, OD meeting AB, AC, and the given circle at P, Q, and R.

Then a circle described with centre O *and radius* OP *will touch* AB, AC *and the given circle.*

For since O is the centre of the ⊙ MND,
∴ OM = ON = OD.
But PM = QN = RD; *Constr.*
∴ OP = OQ = OR.

∴ a circle described with centre O, and radius OP, will pass through Q and R.

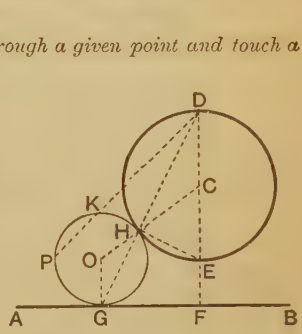
And since the ∠[s] at M and N are rt. angles, III. 18.
∴ the ∠[s] at P and Q are rt. angles; I. 29.
∴ the ⊙ PQR touches AB and AC.

And since R, the point in which the circles meet, is on the line of centres OD,

∴ the ⊙ PQR touches the given circle. Q.E.F.

NOTE. There will be two solutions of this problem, since two circles may be drawn to touch EF, GH and to pass through D.

25. *To describe a circle to pass through a given point and touch a given straight line and a given circle.*



Let P be the given point, AB the given st. line, and DHE the given circle, of which C is the centre.

It is required to describe a circle to pass through P, *and to touch* AB *and the* ⊙ DHE.

Through C draw DCEF perp. to AB, cutting the circle at the points D and E, of which E is between C and AB.

Join DP;

and by describing a circle through F, E, and P, find a point K in DP (or DP produced) such that the rect. DE, DF = the rect. DK, DP.

Describe a circle to pass through P, K, and touch AB: Ex. 21, p. 253.

This circle shall also touch the given ⊙ DHE.

For let G be the point at which this circle touches AB.

Join DG, cutting the given circle DHE at H.

Join HE.

Then the ∠ DHE is a rt. angle, being in a semicircle, III. 31.

also the angle at F is a rt. angle; *Constr.*

∴ the points E, F, G, H are concyclic:

∴ the rect. DE, DF = the rect. DH, DG: III. 36.

but the rect. DE, DF = the rect. DK, DP: *Constr.*

∴ the rect. DH, DG = the rect. DK, DP:

∴ the point H is on the ⊙ PKG.

Let O be the centre of the ⊙ PHG.

Join OG, OH, CH.

Then OG and DF are par[l], since they are both perp. to AB;

and DG meets them.

∴ the ∠ OGD = the ∠ GDC. I. 29.

But since OG = OH, and CD = CH,

∴ the ∠ OGH = the ∠ OHG; and the ∠ CDH = the ∠ CHD:

∴ the ∠ OHG = the ∠ CHD;

∴ OH and CH are in one st. line.

∴ the ⊙ PHG touches the given ⊙ DHE. Q.E.F.

NOTES. (i) Since two circles may be drawn to pass through P, K and to touch AB, it follows that there will be two solutions of the present problem.

(ii) Two more solutions may be obtained by joining PE, and proceeding as before.

The student should examine the nature of the contact between the circles in each case.

26. Describe a circle to pass through a given point, to touch a given straight line, and to have its centre on another given straight line.

27. Describe a circle to pass through a given point, to touch a given circle, and to have its centre on a given straight line.

28. Describe a circle to pass through two given points, and to intercept an arc of given length on a given circle.

29. Describe a circle to touch a given circle and a given straight line at a given point.

30. Describe a circle to touch two given circles and a given straight line.

V. ON MAXIMA AND MINIMA.

We gather from the Theory of Loci that the position of an angle, line or figure is capable under suitable conditions of gradual change; and it is usually found that change of *position* involves a corresponding and gradual change of *magnitude*.

Under these circumstances we may be required to note if any situations exist at which the magnitude in question, after increasing, begins to decrease; or after decreasing, to increase: in such situations the magnitude is said to have reached a **Maximum** or a **Minimum** value: for in the former case it is greater, and in the latter case less than in adjacent situations on either side. In the geometry of the circle and straight line we only meet with such cases of continuous change as admit of *one* transition from an increasing to a decreasing state—or vice versâ—so that in all the problems with which we have to deal (where a single circle is involved) there can be only one Maximum and one Minimum—the Maximum being the greatest, and the Minimum being the least value that the variable magnitude is capable of taking.

Thus a variable geometrical magnitude reaches its maximum or minimum value at a *turning point*, towards which the magnitude may mount or descend from either side: it is natural therefore to expect a maximum or minimum value to occur when, in the course of its change, the magnitude assumes a *symmetrical* form or position; and this is usually found to be the case.

This general connection between a symmetrical form or position and a maximum or minimum value is not exact enough to constitute a *proof* in any particular problem; but by means of it a situation is suggested, which on further examination may be shewn to give the maximum or minimum value sought for.

For example, suppose it is required

to determine the greatest straight line that may be drawn perpendicular to the chord of a segment of a circle and intercepted between the chord and the arc:

we immediately anticipate that the greatest perpendicular is that which occupies a *symmetrical* position in the figure, namely the perpendicular which passes through the middle point of the chord; and on further examination this may be proved to be the case by means of I. 19, and I. 34.

Again we are able to find at what point a geometrical magnitude, varying under certain conditions, assumes its Maximum or Minimum value, if we can discover a construction for drawing the magnitude so that it may have an *assigned* value: for we may then examine between what limits the assigned value must lie in order that the construction may be possible; and the higher or lower limit will give the Maximum or Minimum sought for.

It was pointed out in the chapter on the Intersection of Loci, [see page 125] that if under certain conditions existing among the data, *two* solutions of a problem are possible, and under other conditions, *no* solution exists, there will always be some intermediate condition under which *one* and *only one* distinct solution is possible.

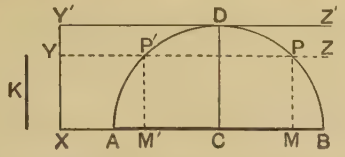
Under these circumstances this single or limiting solution will always be found to correspond to the maximum or minimum value of the magnitude to be constructed.

1. For example, suppose it is required

to divide a given straight line so that the rectangle contained by the two segments may be a maximum.

We may first attempt to divide the given straight line so that the rectangle contained by its segments may have a *given* area—that is, be equal to the square on a given straight line.

Let AB be the given straight line, and K the side of the given square.



It is required to divide the st. line AB *at a point* M, *so that the rect.* AM, MB *may be equal to the sq. on* K.

Adopting a construction suggested by II. 14,

describe a semicircle on AB; and at any point X in AB, or AB produced, draw XY perp. to AB, and equal to K.

Through Y draw YZ par^{l} to AB, to meet the arc of the semicircle at P.

Then if the perp. PM is drawn to AB, it may be shewn after the manner of II. 14, or by III. 35 that

the rect. AM, MB = the sq. on PM
= the sq. on K.

So that the rectangle AM, MB increases as K increases.

Now if K is less than the radius CD, then YZ will meet the arc of the semicircle in two points P, P'; and it follows that AB may be divided at *two* points, so that the rectangle contained by its segments may be equal to the square on K. If K increases, the st. line YZ will recede from AB, and the points of intersection P, P' will continually approach one another; until, when K is equal to the radius CD, the st. line YZ (now in the position Y'Z') will meet the arc in *two coincident points*, that is, will touch the semicircle at D; and there will be only *one* solution of the problem.

If K is greater than CD, the straight line YZ will not meet the semicircle, and the problem is impossible.

Hence the greatest length that K may have, in order that the construction may be possible, is the radius CD.

∴ the rect. AM, MB is a maximum, when it is equal to the square on CD;

that is, when PM coincides with CD, and consequently when M is the middle point of AB.

NOTE. The special feature to be noticed in this problem is that the maximum is found at the transitional point between *two* solutions and *no* solution; that is, when the two solutions coincide and become identical.

The following example illustrates the same point.

2. *To find at what point in a given straight line the angle subtended by the line joining two given points, which are on the same side of the given straight line, is a maximum.*

Let CD be the given st. line, and A, B the given points on the same side of CD.

It is required to find at what point in CD *the angle subtended by the st. line* AB *is a maximum.*

First determine at what point in CD, the st. line AB subtends a *given* angle.

This is done as follows:—

On AB describe a segment of a circle containing an angle equal to the given angle. III. 33.

If the arc of this segment intersects CD, *two* points in CD are found at which AB subtends the given angle: but if the arc does not meet CD, *no* solution is given.

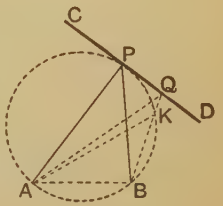
In accordance with the principles explained above, we expect that a maximum angle is determined at the limiting position; that is, when the arc touches CD, or meets it at two coincident points.

[See page 231.]

This we may prove to be the case.

Describe a circle to pass through A and B, and to touch the st. line CD.

[Ex. 21, p. 253.]



Let P be the point of contact.

Then shall the ∠ APB be greater than any other angle subtended by AB at a point in CD on the same side of AB as P.

For take Q, any other point in CD, on the same side of AB as P;

and join AQ, QB.

Since Q is a point in the tangent other than the point of contact, it must be without the circle;

∴ either BQ or AQ must meet the arc of the segment APB.

Let BQ meet the arc at K: join AK.

Then the ∠ APB = the ∠ AKB, in the same segment:

but the ext. ∠ AKB is greater than the int. opp. ∠ AQB.

∴ the ∠ APB is greater than AQB.

Similarly the ∠ APB may be shewn to be greater than any other angle subtended by AB at a point in CD on the same side of AB:

that is, the ∠ APB is the greatest of all such angles. Q.E.D.

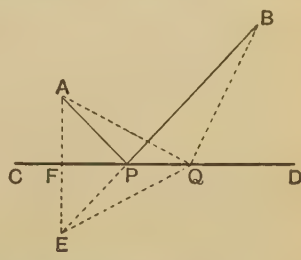
NOTE. Two circles may be described to pass through A and B, and to touch CD, the points of contact being on opposite sides of AB;

hence two points in CD may be found such that the angle subtended by AB at each of them is greater than the angle subtended at any other point in CD *on the same side of* AB.

We add two more examples of considerable importance.

3. *In a straight line of indefinite length find a point such that the sum of its distances from two given points, on the same side of the given line, shall be a minimum.*

Let CD be the given st. line of indefinite length, and A, B the given points on the same side of CD.



It is required to find a point P *in* CD, *such that the sum of* AP, PB *is a minimum.*

Draw AF perp. to CD; and produce AF to E, making FE equal to AF.

Join EB, cutting CD at P.

Join AP, PB.

Then of all lines drawn from A *and* B *to a point in* CD, *the sum of* AP, PB *shall be the least.*

For, let Q be any other point in CD.

Join AQ, BQ, EQ.

Now in the $\triangle^s$ AFP, EFP,

Because { AF = EF, *Constr.*
and FP is common;
and the $\angle$ AFP = the $\angle$ EFP, being rt. angles.

$\therefore$ AP = EP. I. 4.

Similarly it may be shewn that

AQ = EQ.

Now in the $\triangle$ EQB, the two sides EQ, QB are together greater than EB;

hence, AQ, QB are together greater than EB,

that is, greater than AP, PB.

Similarly the sum of the st. lines drawn from A and B to any other point in CD may be shewn to be greater than AP, PB.

$\therefore$ the sum of AP, PB is a minimum.

Q. E. D.

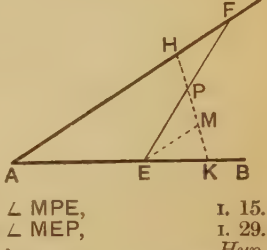
NOTE. It follows from the above proof that

the $\angle$ APF = the $\angle$ EPF I. 4.

= the $\angle$ BPD. I. 15.

Thus the sum of AP, PB is a minimum, when these lines are *equally inclined* to CD.

4. *Given two intersecting straight lines* AB, AC, *and a point* P *between them; shew that of all straight lines which pass through* P *and are terminated by* AB, AC, *that which is bisected at* P *cuts off the triangle of minimum area.*



Let EF be the st. line, terminated by AB, AC, which is bisected at P.

Then the △ FAE *shall be of minimum area.*

For let HK be any other st. line passing through P.

Through E draw EM par^{l} to AC.

Then in the △^{s} HPF, MPE,

Because { the ∠ HPF = the ∠ MPE, I. 15.
and the ∠ HFP = the ∠ MEP, I. 29.
and FP = EP; *Hyp.*
∴ the △ HPF = the △ MPE. I. 26, Cor.

But the △ MPE is less than the △ KPE;
∴ the △ HPF is less than the △ KPE:
to each add the fig. AHPE;
then the △ FAE is less than the △ HAK.

Similarly it may be shewn that the △ FAE is less than any other triangle formed by drawing a st. line through P:
that is, the △ FAE is a minimum.

Examples.

1. Two sides of a triangle are given in length; how must they be placed in order that the area of the triangle may be a maximum?

2. *Of all triangles of given base and area, the isosceles is that which has the least perimeter.*

3. Given the base and vertical angle of a triangle; construct it so that its area may be a maximum.

4. Find a point in a given straight line such that the tangents drawn from it to a given circle contain the greatest angle possible.

5. A straight rod slips between two straight rulers placed at right angles to one another; in what position is the triangle intercepted between the rulers and rod a maximum?

Thurs.

6. Divide a given straight line into two parts, so that the sum of the squares on the segments

(i) may be equal to a given square;

(ii) may be a minimum.

7. Through a point of intersection of two circles draw a straight line terminated by the circumferences,

(i) so that it may be of given length;

(ii) so that it may be a maximum.

8. Two tangents to a circle cut one another at right angles; find the point on the intercepted arc such that the sum of the perpendiculars drawn from it to the tangents may be a minimum.

9. Straight lines are drawn from two given points to meet one another on the convex circumference of a given circle: prove that their sum is a minimum when they make equal angles with the tangent at the point of intersection.

10. Of all triangles of given vertical angle and altitude, that which is isosceles has the least area.

11. Two straight lines CA, CB of indefinite length are drawn from the centre of a circle to meet the circumference at A and B; then of all tangents that may be drawn to the circle at points on the arc AB, that whose intercept is bisected at the point of contact cuts off the triangle of minimum area.

12. Given two intersecting tangents to a circle, draw a tangent to the convex arc so that the triangle formed by it and the given tangents may be of maximum area.

13. Of all triangles of given base and area, that which is isosceles has the greatest vertical angle.

14. Find a point on the circumference of a circle at which the straight line joining two given points (of which both are within, or both without the circle) subtends the greatest angle.

15. A bridge consists of three arches, whose spans are 49 ft., 32 ft. and 49 ft. respectively: shew that the point on either bank of the river at which the middle arch subtends the greatest angle is 63 feet distant from the bridge.

16. From a given point P without a circle whose centre is C, draw a straight line to cut the circumference at A and B, so that the triangle ACB may be of maximum area.

17. Shew that the greatest rectangle which can be inscribed in a circle is a square.

18. A and B are two fixed points without a circle: find a point P on the circumference, such that the sum of the squares on AP, PB may be a minimum. [See p. 161. Ex. 24.]

19. A segment of a circle is described on the chord AB: find a point C on its arc so that the sum of AC, BC may be a maximum.

20. *Of all triangles that can be inscribed in a circle that which has the greatest perimeter is equilateral.*

21. *Of all triangles that can be inscribed in a given circle that which has the greatest area is equilateral.*

22. *Of all triangles that can be inscribed in a given triangle that which has the least perimeter is the triangle formed by joining the feet of the perpendiculars drawn from the vertices on opposite sides.*

23. Of all rectangles of given area, the square has the least perimeter.

24. Describe the triangle of maximum area, having its angles equal to those of a given triangle, and its sides passing through three given points.

VI. HARDER MISCELLANEOUS EXAMPLES.

1. AB is a diameter of a given circle; and AC, BD, two chords on the same side of AB, intersect at E: shew that the circle which passes through D, E, C cuts the given circle orthogonally.

2. Two circles whose centres are C and D intersect at A and B; and a straight line PAQ is drawn through A and terminated by the circumferences: prove that

(i) the angle PBQ = the angle CAD

(ii) the angle BPC = the angle BQD.

3. Two chords AB, CD of a circle whose centre is O intersect at right angles at P: shew that

(i) $PA^2 + PB^2 + PC^2 + PD^2 = 4$ (radius)^2.

(ii) $AB^2 + CD^2 + 4OP^2 = 8$ (radius)^2.

4. Two parallel tangents to a circle intercept on any third tangent a portion which is so divided at its point of contact that the rectangle contained by its two parts is equal to the square on the radius.

5. Two equal circles move between two straight lines placed at right angles, so that each straight line is touched by one circle, and the two circles touch one another: find the locus of the point of contact.

6. AB is a given diameter of a circle, and CD is any parallel chord: if any point X in AB is joined to the extremities of CD, shew that

$$XC^2 + XD^2 = XA^2 + XB^2.$$

7. PQ is a fixed chord in a circle, and PX, QY any two parallel chords through P and Q; shew that XY touches a fixed concentric circle.

8. Two equal circles intersect at A and B; and from C, any point on the circumference of one of them, a perpendicular is drawn to AB, meeting the other circle at O and O'; shew that either O or O' is the orthocentre of the triangle ABC. Distinguish between the two cases.

9. Three equal circles pass through the same point A, and their other points of intersection are B, C, D: shew that of the four points A, B, C, D, each is the orthocentre of the triangle formed by joining the other three.

10. From a given point without a circle draw a straight line to the concave circumference so as to be bisected by the convex circumference. When is this problem impossible?

11. Draw a straight line cutting two concentric circles so that the chord intercepted by the circumference of the greater circle may be double of the chord intercepted by the less.

12. ABC is a triangle inscribed in a circle, and A', B', C' are the middle points of the arcs subtended by the sides (remote from the opposite vertices): find the relation between the angles of the two triangles ABC, A'B'C'; and prove that the pedal triangle of A'B'C' is equiangular to the triangle ABC.

13. The opposite sides of a quadrilateral inscribed in a circle are produced to meet: shew that the bisectors of the two angles so formed are perpendicular to one another.

14. If a quadrilateral can have one circle inscribed in it, and another circumscribed about it; shew that the straight lines joining the opposite points of contact of the inscribed circle are perpendicular to one another.

15. Given the base of a triangle and the sum of the remaining sides; find the locus of the foot of the perpendicular from one extremity of the base on the bisector of the exterior vertical angle.

16. Two circles touch each other at C, and straight lines are drawn through C at right angles to one another, meeting the circles at P, P' and Q, Q' respectively: if the straight line which joins the centres is terminated by the circumferences at A and A', shew that

$$P'P^2 + Q'Q^2 = A'A^2.$$

17. Two circles cut one another orthogonally at A and B; P is any point on the arc of one circle intercepted by the other, and PA, PB are produced to meet the circumference of the second circle at C and D: shew that CD is a diameter.

18. ABC is a triangle, and from any point P perpendiculars PD, PE, PF are drawn to the sides: if S_1, S_2, S_3 are the centres of the circles circumscribed about the triangles EPF, FPD, DPE, shew that the triangle $S_1S_2S_3$ is equiangular to the triangle ABC, and that the sides of the one are respectively half of the sides of the other.

19. Two tangents PA, PB are drawn from an external point P to a given circle, and C is the middle point of the chord of contact AB; if XY is any chord through P, shew that AB bisects the angle XCY.

20. Given the sum of two straight lines and the rectangle contained by them (equal to a given square): find the lines.

21. Given the sum of the squares on two straight lines and the rectangle contained by them: find the lines.

22. Given the sum of two straight lines and the sum of the squares on them: find the lines.

23. Given the difference between two straight lines, and the rectangle contained by them: find the lines.

24. Given the sum or difference of two straight lines and the difference of their squares: find the lines.

25. ABC is a triangle, and the internal and external bisectors of the angle A meet BC, and BC produced, at P and P′: if O is the middle point of PP′, shew that OA is a tangent to the circle circumscribed about the triangle ABC.

26. ABC is a triangle, and from P, any point on the circumference of the circle circumscribed about it, perpendiculars are drawn to the sides BC, CA, AB meeting the circle again in A′, B′, C′; prove that

(i) the triangle A′B′C′ is identically equal to the triangle ABC.

(ii) AA′, BB′, CC′ are parallel.

27. Two equal circles intersect at fixed points A and B, and from any point in AB a perpendicular is drawn to meet the circumferences on the same side of AB at P and Q: shew that PQ is of constant length.

28. The straight lines which join the vertices of a triangle to the centre of its circumscribed circle, are perpendicular respectively to the sides of the pedal triangle.

29. P is any point on the circumference of a circle circumscribed about a triangle ABC; and perpendiculars PD, PE are drawn from P to the sides BC, CA. Find the locus of the centre of the circle circumscribed about the triangle PDE.

30. P is any point on the circumference of a circle circumscribed about a triangle ABC: shew that the angle between Simson's Line for the point P and the side BC is equal to the angle between AP and the diameter of the circumscribed circle through A.

31. Shew that the circles circumscribed about the four triangles formed by two pairs of intersecting straight lines meet in a point.

32. Shew that the orthocentres of the four triangles formed by two pairs of intersecting straight lines are collinear.

On the Construction of Triangles.

33. Given the vertical angle, one of the sides containing it, and the length of the perpendicular from the vertex on the base: construct the triangle.

34. Given the feet of the perpendiculars drawn from the vertices on the opposite sides: construct the triangle.

35. Given the base, the altitude, and the radius of the circumscribed circle: construct the triangle.

36. Given the base, the vertical angle, and the sum. of the squares on the sides containing the vertical angle: construct the triangle.

37. Given the base, the altitude and the sum of the squares on the sides containing the vertical angle: construct the triangle.

38. Given the base, the vertical angle, and the difference of the squares on the sides containing the vertical angle: construct the triangle.

39. Given the vertical angle, and the lengths of the two medians drawn from the extremities of the base: construct the triangle.

40. Given the base, the vertical angle, and the difference of the angles at the base: construct the triangle.

41. Given the base, and the position of the bisector of the vertical angle: construct the triangle.

42. Given the base, the vertical angle, and the length of the bisector of the vertical angle: construct the triangle.

43. Given the perpendicular from the vertex on the base, the bisector of the vertical angle, and the median which bisects the base: construct the triangle.

44. Given the bisector of the vertical angle, the median bisecting the base, and the difference of the angles at the base: construct the triangle.

APPENDIX.

ON POLE AND POLAR.

DEFINITIONS.

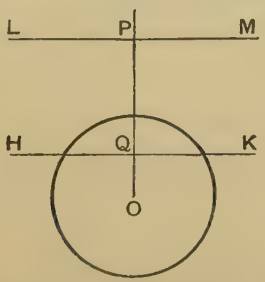
(i) If in any straight line drawn from the centre of a circle two points are taken such that the rectangle contained by their distances from the centre is equal to the square on the radius, each point is said to be the **inverse** of the other.

Thus in the figure given below, if O is the centre of the circle, and if OP . OQ = (radius)^2, then each of the points P and Q is the inverse of the other.

It is clear that if one of these points is within the circle the other must be without it.

(ii) The **polar** of a given point with respect to a given circle is the straight line drawn through the inverse of the given point at right angles to the line which joins the given point to the centre: and with reference to the polar the given point is called the **pole**.

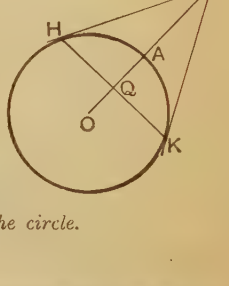
Thus in the adjoining figure, if OP . OQ = (radius)^2, and if through



P and Q, LM and HK are drawn perp. to OP; then HK is the polar of the point P, and P is the pole of the st. line HK: also LM is the polar of the point Q, and Q the pole of LM.

It is clear that the polar of an *external* point must intersect the circle, and that the polar of an *internal* point must fall without it: also that the polar of a point *on the circumference* is the tangent at that point.

1. Now it has been proved [see Ex. 1, page 251] that if from an external point P two tangents PH, PK are drawn to a circle, of which O is the centre, then OP cuts the chord of contact HK at right angles at Q, so that



$$OP \cdot OQ = (\text{radius})^2,$$

$\therefore$ HK is the polar of P with respect to the circle. *Def.* (ii).

Hence we conclude that

The Polar of an external point with reference to a circle is the chord of contact of tangents drawn from the given point to the circle.

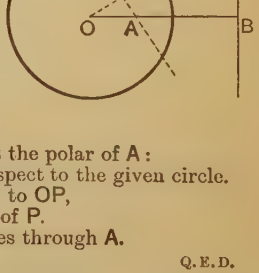
The following Theorem is known as the **Reciprocal Property** of Pole and Polar.

2. *If* A *and* P *are any two points, and if the polar of* A *with respect to any circle passes through* P, *then the polar of* P *must pass through* A.

Let BC be the polar of the point A with respect to a circle whose centre is O, and let BC pass through P.

Then shall the polar of P *pass through* A.

Join OP; and from A draw AQ perp. to OP. We shall shew that AQ is the polar of P.



Now since BC is the polar of A,

$\therefore$ the $\angle$ ABP is a rt. angle; *Def.* (ii), p. i.

and the $\angle$ AQP is a rt. angle: *Constr.*

$\therefore$ the four points A, B, P, Q are concyclic;

$\therefore$ $OQ \cdot OP = OA \cdot OB$ III. 36.

$= (\text{radius})^2$, for CB is the polar of A:

$\therefore$ P and Q are inverse points with respect to the given circle.

And since AQ is perp. to OP,

$\therefore$ AQ is the polar of P.

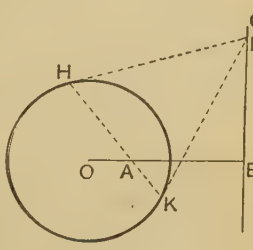
That is, the polar of P passes through A.

Q.E.D.

A similar proof applies to the case when the given point A is without the circle, and the polar BC cuts it.

3. *To prove that the locus of the intersection of tangents drawn to a circle at the extremities of all chords which pass through a given point within the circle is the polar of that point.*

Let A be the given point within the circle. Let HK be *any* chord passing through A; and let the tangents at H and K intersect at P.



It is required to prove that the locus of P *is the polar of the point* A.

I. To shew that P lies on the polar of A.

Since HK is the chord of contact of tangents drawn from P.

∴ HK is the polar of P. Ex. 1, p. ii.

But HK, the polar of P, passes through A;

∴ the polar of A passes through P; Ex. 2, p. ii.

that is, the point P lies on the polar of A.

II. To shew that *any* point on the polar of A satisfies the given conditions.

Let BC be the polar of A, and let P be any point on it.

Draw tangents PH, PK, and let HK be the chord of contact.

Now from Ex. 1, p. ii, we know that the chord of contact HK is the polar of P,

and we also know that the polar of P must pass through A; for P is on BC, the polar of A: Ex. 2, p. ii.

that is, HK passes through A.

∴ P is the point of intersection of tangents drawn at the extremities of a chord passing through A.

From I. and II. we conclude that the required locus is the polar of A.

NOTE. If A is *without* the circle, the theorem demonstrated in Part I. of the above proof still holds good; but the converse theorem in Part II. is not true for *all* points in BC. For if A is without the circle, the polar BC will intersect it; and no point on that part of the polar which is within the circle can be the point of intersection of tangents.

A summary of the results obtained in this Section will be found on the following page.

SUMMARY.

(i) *The Polar of an* external *point with respect to a circle is the chord of contact of tangents drawn from it.*

(ii) *The Polar of an* internal *point is the locus of the intersections of tangents drawn at the extremities of all chords which pass through it.*

(iii) *The polar of a* point on the circumference *is the tangent at that point.*

EXAMPLES ON POLE AND POLAR.

1. *The straight line which joins any two points is the polar with respect to a given circle of the point of intersection of their polars.*

2. *The point of intersection of any two straight lines is the pole of the straight line which joins their poles.*

3. *Find the locus of the poles of all straight lines which pass through a given point.*

4. *Find the locus of the poles, with respect to a given circle, of tangents drawn to a concentric circle.*

5. *If two circles cut one another orthogonally, and* PQ *be any diameter of one of them; shew that the polar of* P *with regard to the other circle passes through* Q.

6. *If two circles cut one another orthogonally, the centre of each circle is the pole of their common chord with respect to the other circle.*

7. *Any two points subtend at the centre of a circle an angle equal to one of the angles formed by the polars of the given points.*

8. O *is the centre of a given circle, and* AB *a fixed straight line.* P *is any point in* AB; *find the locus of the point inverse to* P *with respect to the circle.*

9. *Given a circle, and a fixed point* O *on its circumference:* P *is* any *point on the circle: find the locus of the point inverse to* P *with respect to any circle whose centre is* O.

10. *Given two points* A *and* B, *and a circle whose centre is* O; *shew that the rectangle contained by* OA *and the perpendicular from* B *on the polar of* A *is equal to the rectangle contained by* OB *and the perpendicular from* A *on the polar of* B.

II. ON THE RADICAL AXIS.

1. *To find the locus of points from which the tangents drawn to two given circles are equal.*

Fig. 1.

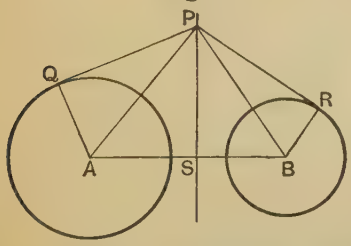
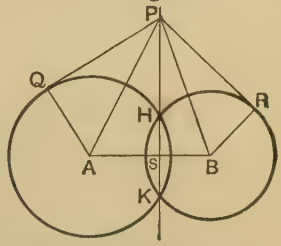


Fig. 2.



Let A and B be the centres of the given circles, whose radii are a and b; and let P be any point such that the tangent PQ drawn to the circle (A) is equal to the tangent PR drawn to the circle (B).

It is required to find the locus of P.

Join PA, PB, AQ, BR, AB; and from P draw PS perp. to AB.

Then because $PQ = PR$, $\therefore\ PQ^2 = PR^2$.

But $PQ^2 = PA^2 - AQ^2$; and $PR^2 = PB^2 - BR^2$: I. 47.

$\therefore\ PA^2 - AQ^2 = PB^2 - BR^2$;

that is, $PS^2 + AS^2 - a^2 = PS^2 + SB^2 - b^2$; I. 47.

or, $AS^2 - a^2 = SB^2 - b^2$.

Hence AB is divided at S, so that $AS^2 - SB^2 = a^2 - b^2$:

$\therefore$ S is a *fixed point.*

Hence all points from which equal tangents can be drawn to the two circles lie on the straight line which cuts AB at rt. angles, so that the difference of the squares on the segments of AB is equal to the difference of the squares on the radii.

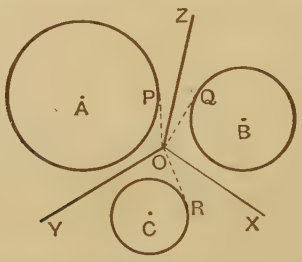
Again, by simply retracing these steps, it may be shewn that in Fig. 1 every point in SP, and in Fig. 2 every point in SP exterior to the circles, is such that tangents drawn from it to the two circles are equal.

Hence we conclude that in Fig. 1 the whole line SP is the required locus, and in Fig. 2 that part of SP which is without the circles.

In either case SP is said to be the **Radical Axis** of the two circles.

COROLLARY. *If the circles cut one another as in* Fig. 2, *it is clear that the Radical Axis is identical with the straight line which passes through the points of intersection of the circles;* for it follows readily from III. 36 that tangents drawn to two intersecting circles from any point in the common chord produced are equal.

2. *The Radical Axes of three circles taken in pairs are concurrent.*



Let there be three circles whose centres are A, B, C.
Let OZ be the radical axis of the ⊙ˢ (A) and (B);
and OY the Radical Axis of the ⊙ˢ (A) and (C), O being the point of their intersection.

Then shall the radical axis of the ⊙ˢ (B) *and* (C) *pass through* O.

It will be found that the point O is either *without* or *within all the circles.*

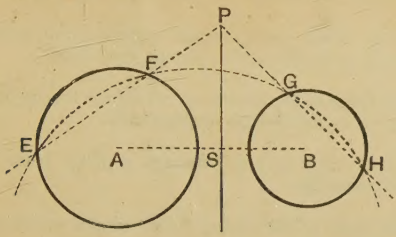
I. When O is without the circles.
From O draw OP, OQ, OR tangents to the ⊙ˢ (A), (B), (C).

Then because O is a point on the radical axis of (A) and (B); *Hyp.*
∴ OP=OQ.
And because O is a point on the radical axis of (A) and (C), *Hyp.*
∴ OP=OR;
∴ OQ=OR;
∴ O is a point on the radical axis of (B) and (C),
i.e. the radical axis of (B) and (C) passes through O.

II. If the circles intersect in such a way that O is within them all;
the radical axes are then the common chords of the three circles taken two and two; and it is required to prove that these common chords are concurrent. This may be shewn indirectly by III. 35.

DEFINITION. The point of intersection of the radical axes of three circles taken in pairs is called the **radical centre.**

3. *To draw the radical axis of two given circles.*



Let A and B be the centres of the given circles.

It is required to draw their radical axis.

If the given circles intersect, then the st. line drawn through their points of intersection will be the radical axis. [Ex. 1, Cor. p. vi.]

But if the given circles do not intersect,

describe any circle so as to cut them in E, F and G, H.

Join EF and HG, and produce them to meet in P.

Join AB; and from P draw PS perp. to AB.

Then PS *shall be the radical axis of the* $\odot^s$ (A), (B).

DEFINITION. If each pair of circles in a given system have the same radical axis, the circles are said to be **co-axal.**

EXAMPLES.

1. *Shew that the radical axis of two circles bisects any one of their common tangents.*

2. *If tangents are drawn to two circles from any point on their radical axis; shew that a circle described with this point as centre and any one of the tangents as radius, cuts both the given circles orthogonally.*

3. O *is the radical centre of three circles, and from* O *a tangent* OT *is drawn to any one of them: shew that a circle whose centre is* O *and radius* OT *cuts all the given circles orthogonally.*

4. *If three circles touch one another, taken two and two, shew that their common tangents at the points of contact are concurrent.*

5. *If circles are described on the three sides of a triangle as diameter, their radical centre is the orthocentre of the triangle.*

6. *All circles which pass through a fixed point and cut a given circle orthogonally, pass through a second fixed point.*

7. *Find the locus of the centres of all circles which pass through a given point and cut a given circle orthogonally.*

8. *Describe a circle to pass through a given point and cut two given circles orthogonally.*

9. *Find the locus of the centres of all circles which cut two given circles orthogonally.*

10. *Describe a circle to pass through a given point and cut two given circles orthogonally.*

11. *The difference of the squares on the tangents drawn from any point to two circles is equal to twice the rectangle contained by the straight line joining their centres and the perpendicular from the given point on their radical axis.*

12. *In a system of co-axal circles which do not intersect, any point is taken on the radical axis; shew that a circle described from this point as centre with radius equal to the tangent drawn from it to any one of the circles, will meet the line of centres in two* fixed *points.*

[*These fixed points are called the* **Limiting Points** *of the system.*]